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The remainder in Tauberian theorems

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1. Introduction

In his paper "Tauberian Theorems" of 1932, N. WIENER stated a Tauberian theorem which was sufficiently general to include many of the various earlier Tauberian theorems. In a form convenient for our purposes this theorem may be stated as follows (cf. PITT [2]).

Wiener's general Tauberian theorem

$K(x)$ and $\Phi^*(x)$ are functions of the real variable x, and we suppose

$$\Phi^*(x) \text{ real and bounded}$$

$$\Phi^*(x+h)-\Phi^*(x)\geq w(h), \qquad w(h)\to 0 \text{ when } h\to +0.$$

$$K(x)\in L,\ k(t)=\int_{-\infty}^{\infty} K(u)e^{itu}\,du\neq 0 \text{ for } t \text{ real.}$$

Then

$$\lim_{x\to\infty}\int_{-\infty}^{\infty}\Phi^*(x-u)K(u)\,du=A\int_{-\infty}^{\infty}K(u)\,du$$

implies

$$\lim_{x\to\infty}\Phi^*(x)=A.$$

Let

$$\Psi^*(x)=\int_{-\infty}^{\infty}\Phi^*(x-u)K(u)\,du.$$

Then a Tauberian theorem yields an asymptotic estimation of the function $\Phi^*(x)$, if we know the asymptotic behaviour of the function $\Psi^*(x)$. The question arises whether it is possible to estimate the "remainder" $\Phi(x)=\Phi^*(x)-A$ when we know the behaviour of the analoguous "remainder" $\Psi(x)=\Psi^*(x)-A\int_{-\infty}^{\infty}K(x)\,dx$. Wiener's Tauberian theorem merely changes the imposed condition $\Phi(x)=O(1)$ to $\Phi(x)=o(1)$, $x\to\infty$. Wiener himself considered, in the paper quoted, that it is impossible to reach better results with his methods.

Since then, however, theorems have appeared which prove that under more restricted conditions on the kernel $K(x)$, it is actually possible to estimate the remainder $\Phi(x)$. (See BEURLING [1] page 22.) The present paper is confined to Tauberian relations for which such an estimation of the remainder is possible.

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