Communicated 8 April 1953 by ARNE BEURLING and ÅKE PLEIJEL

Fourier transforms of the class $\mathfrak{L}_p$

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It is well known that the theorem of RIESZ-FISCHER and the theorem of PLANCHEREL, dealing with Fourier transforms of the classes $\mathfrak{L}_2$ on the circle and line, respectively, have analogues for other classes $\mathfrak{L}_p\,(1 < p < 2)$. Thus the theorem of YOUNG-HAUSDORFF states that if f is any function on $[0, 2\pi]$ such that $\int\limits_0^{2\pi} |f(x)|^p\,dx < \infty$, then the numbers $c_n = \frac{1}{2\pi}\int\limits_0^{2\pi} e^{inx} f(x)\,dx$ have the property that

$$\sum_{n=-\infty}^{\infty} |c_n|^{p'} < \infty, \tag{1}$$

where $p' = \frac{p}{p-1}$, and

$$\left[\frac{1}{2\pi}\int\limits_0^{2\pi} |f(x)|^p\,dx\right]^{\frac{1}{p}} \geq \left[\sum_{n=-\infty}^{\infty} |c_n|^{p'}\right]^{\frac{1}{p'}} \tag{2}$$

(See for example [5], pp. 189–202.) An analogous theorem, proved by TITCHMARSH (see [3], pp. 96–107), shows that every function f in $\mathfrak{L}_p(-\infty, \infty)$ admits a Fourier transform of class $\mathfrak{L}_{p'}$ with norm in $\mathfrak{L}_{p'}(-\infty, \infty)$ majorized by a constant times the $\mathfrak{L}_p$ norm of f. For both of these cases, examples can be given to show that not all sequences of class $l_{p'}$ or functions of class $\mathfrak{L}_{p'}$ can be obtained as Fourier transforms of the class $\mathfrak{L}_p$. (See [5], p. 190, and [3], pp. 111–112.) It is the purpose of the present note to show that this phenomenon must appear for all infinite locally compact Abelian groups.

Throughout the present note, let G stand for a locally compact Abelian group. Integration with regard to a suitably normalized Haar measure on G is indicated by expressions such as

$$\int\limits_G f(x)\,dx. \tag{3}$$

For all numbers $r \geq 1$, the symbol $\mathfrak{L}_r$ denotes the space of all complex-valued Haar measurable functions f such that

$$\int\limits_G |f(x)|^r\,dx < \infty, \tag{4}$$