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A study of certain Green's functions with applications in the theory of vibrating membranes

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Introduction. The first part of this paper contains estimations for $\varkappa \to +\infty$ of the Green's functions which satisfy the equation

$$\left(\frac{\partial}{\partial x^1}\right)^2 u + \left(\frac{\partial}{\partial x^2}\right)^2 u - \varkappa^2 u = 0 \tag{1}$$

and Dirichlet's or Neumann's boundary conditions. A similar investigation in the theory of Laplace's equation was recently carried out in collaboration with T. GANELIUS. A previous paper on Green's functions of the biharmonic equation has been published in the proceedings of the Symposium on Spectral Theory and Differential Problems in Stillwater, Oklahoma, 1951.

In the second part of the paper, eigenvalue problems of vibrating membranes are studied by CARLEMAN's methods [1]. By the help of the results of part I certain theorems on the asymptotic behaviour of the eigenvalues and eigenfunctions are obtained.

In order to simplify the exposition, only membranes with infinitely differentiable boundaries are being considered.

Part I. Estimates for the Green's functions.

1. The equation (1), viz. $\Delta u - \varkappa^2 u = 0$, is considered in an open, bounded and simply connected domain V of the cartesian $x^1 x^2$-plane. The boundary S of V is given by equations $x^i = y^i(s)$, $i = 1, 2$, in which s is the arc-length of the boundary, measured in the counter-clock-wise sense, and $y^i(s)$ are infinitely differentiable functions. S also denotes the total length of the boundary. The distance from a point $x = (x^1, x^2)$ to S is n, this distance being positive when x belongs to V. The letter n also denotes the inner normal of S; $n_s = n_y$ is the normal at the point $y(s) = (y^1(s), y^2(s))$.

The equation (1) has the elementary solution $\frac{1}{2\pi} K_0(\varkappa r)$, where K_0 is the Bessel K_0-function and $r = r_{x_1 x_2}$ is the distance between $x_1 = (x_1^1, x_1^2)$ and $x_2 = (x_2^1, x_2^2)$. We assume $\varkappa > 0$.