ARKIV FÖR MATEMATIK Band 2 nr 28

Communicated 8 April 1953 by ARNE BEURLING and ÅKE PLEIJEL

On a class of normed rings[1]

By J. WERMER

Introduction

A. BEURLING, in [1], introduced the class of normed rings L'_σ with the norm $\|f\| = \int_{-\infty}^{\infty} |f(x)| \sigma(x) dx$, $\sigma(x)$ being a weight function such that $\sigma(x+y) \leq \leq \sigma(x)\sigma(y)$. This paper has its origin in Prof. Beurling's suggestion to use the methods and results of [1] in studying a more general class of spaces. The writer wishes to take this opportunity to express his sincere thanks to Prof. Beurling for his stimulating advice and kind encouragement

We are concerned with certain spaces of functions defined on the real line. Let L be a Banach space of functions summable on $(-\infty, \infty)$, and let $\bar{L}$ consist of all φ with $\int_{-\infty}^{\infty} |f(x)| \cdot |\varphi(x)| dx < \infty$ if $f \in L$. We shall use the following notation: if $f, g \in L$, $f * g(x) = \int_{-\infty}^{\infty} f(x-y) g(y) dy$, while if $f \in \bar{L}$, $\varphi \in L$, $f * \varphi(x) = = \int_{-\infty}^{\infty} f(y-x) \varphi(y) dy$. We shall consider the following conditions on L:

(1) L contains the characteristic functions of all finite intervals.

(2) If $f \in L$ and $g \in L$ and $|f(x)| = |g(x)|$ a. e., then $\|f\| = \|g\|$.

(3) If for a measurable function f we have $\int_{-\infty}^{\infty} |f(x)| \cdot |\psi(x)| dx < \infty$, for all ψ in $\bar{L}$, then $f \in L$.

(4) Every bounded linear functional α on L is of the form $\alpha(f) = \int_{-\infty}^{\infty} f(x) \varphi(x) dx$, where $\varphi \in \bar{L}$, for all f in L; conversely, every φ in $\bar{L}$ defines a bounded functional in this way.

(5) The translation operator $T_\tau : T_\tau f(x) = f(x-\tau)$ is bounded on L for each real τ.

(6) L is a normed ring under convolution, i. e., if $f, g \in L$, then $f \star g \in L$ and $\|f \star g\| \leq k \|f\| \cdot \|g\|$, where k is a constant.

---

[1] Most of this paper forms part of the author's dissertation (Harvard, 1951).