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On the exceptional points of cubic curves

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§ 1

Introduction

1. If the curve

$$y^2 = x^3 - Ax - B \qquad (4A^3 - 27B^2 \neq 0) \tag{1}$$

is represented by the elliptic $\wp$-function with the invariants $4A$ and $4B$ and a primitive pair of periods ω, ω':

$$x = \wp(u); \quad y = \tfrac{1}{2}\wp'(u),$$

a point $(x; y)$ on (1) may be called *the point* u, where u is determined mod ω, ω'.

If the points u_1, u_2, u_3 lie on a straight line, we have

$$u_1 + u_2 + u_3 \equiv 0 \qquad (\text{mod } \omega, \omega').$$

It follows that the tangent in the point u cuts the curve in $-2u$. If the number u is commensurable with a period, and if n is the smallest natural number that makes nu a period, then u is called an *exceptional point of order* n; this notion has been introduced by NAGELL [11]. The point of order 1 is the infinite point of inflexion, the points of order 2 are given by $y = 0$, and the points of order 3 are the finite points of inflexion.

Now suppose that A and B belong to a field Ω. Then u is said to be a *point in* Ω, if its coordinates belong to this field. If u_1 and u_2 are exceptional points in Ω, the same is true of $u_1 + u_2$, and in this way the exceptional points in Ω form an Abelian group, the *exceptional group in* Ω on the curve (1) (see CHÂTELET [17]). If Ω is an algebraic field, it follows from a theorem due to WEIL [16] that this group is finite. If p is a prime, the group contains at most two independent elements of order p, since there are only two independent periods (see BILLING [1], p. 29); consequently a group of order

$$p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r},$$