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On the propagation of analyticity of solutions of differential equations with constant coefficients

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1. Introduction

Let $P(D)$ be a partial differential operator with constant complex coefficients, let Ω be an open set in R^n, and write

$$\Omega_d = \{x;\ x \in \Omega,\ x_n > d\}.$$

If E and F are sets, we let $E \setminus F$ denote the set $E \cap \complement F$. By $C^\infty(\Omega)$ we denote the set of infinitely differentiable complex valued functions in Ω. The following theorem is due to John [5] and Malgrange [6] (see also Hörmander [4], Ch. III, VIII).

Theorem 1. *Let the distribution u in Ω satisfy the equation $P(D)u = f$, where $f \in C^\infty(\Omega_d)$, and assume that $u \in C^\infty(\Omega_d \setminus F)$, where F is a compact subset of Ω, and d is a real number. Then $u \in C^\infty(\Omega_d)$.*

The main purpose of this paper is to prove the analogous result with analyticity instead of infinite differentiability, i.e.

Theorem 2. *Assume in addition to the hypotheses of Theorem 1 that u is real analytic in $\Omega_d \setminus F$ and that f is real analytic in Ω_d. Then u is real analytic in Ω_d.*

We also prove a more general result involving classes of C^∞ functions. Such classes are defined as follows. If $L = \{L_k\}_{k=1}^{\infty}$ is an increasing sequence of positive numbers and Ω an open subset of R^n, we denote by $C^L(\Omega) = C^L$ the set of functions $f \in C^\infty(\Omega)$ such that to every compact set $F \subset \Omega$ there exists a constant C such that

$$|D^\alpha f(x)| \leqslant C^k L_k^k, \quad \text{if} \quad |\alpha| = k, x \in F, \quad k = 1, 2, \ldots.$$

Here D^α denotes $(\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \Sigma \alpha_j$. Note that $f \in C^L(\Omega)$, if $f \in C^L$ in some neighbourhood of every point in Ω. In fact this follows by applying the Borel-Lebesgue lemma. If $L_k = k$ for every k, the class $C^L(\Omega)$ is equal to the class $A(\Omega)$ of all real analytic functions on Ω. Here we shall only consider classes which contain $A(\Omega)$. Every such class can be defined by a sequence satisfying

$$L_k \geqslant k \quad (k = 1, 2, \ldots). \tag{1.1}$$

Definition. *We say that the increasing sequence L is affine invariant, if for any positive integers a and b there exists a constant C such that $C^{-1} L_k \leqslant L_{ak+b} \leqslant C L_k$ for every k.*