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Compact linear mappings between interpolation spaces

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Introduction

Let L^p denote the space of all (equivalence classes of) functions f defined on some subset Ω of the ν-dimensional euclidean space R^ν and such that f is measurable and

$$\int_\Omega |f(x)|^p\,dx < \infty, \quad dx = dx_1 \dots dx_\nu.$$

The well-known M. Riesz theorem states in particular that, if T is a linear operator which maps L^{p_j} continuously into L^{q_j} $(j=0,1)$, then T maps L^p continuously into L^q, where p and q are given by

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}, \quad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1} \quad (0<\theta<1).$$

If in addition $T: L^{p_0} \to L^{q_0}$ is a compact mapping, it was proved by Krasnoselski [2] (cf. also Cotlar [1], where a similar, slightly weaker result is established) that the mapping $T: L^p \to L^q$ also is compact. J. L. Lions (personal communication) posed the problem if this theorem holds true if we replace L^{p_0}, L^{p_1} and L^{q_0}, L^{q_1} by more general interpolation pairs A_0, A_1 and E_0, E_1, respectively, of Banach spaces and L^p and L^q by interpolation spaces A_θ and E_θ of exponent θ with respect to these pairs. We shall prove here that this question can be answered in the affirmative as soon as the interpolation pair E_0, E_1 satisfies a certain approximation hypothesis, a special case of which was already considered by Lions [3] for other purposes. The approximation hypothesis is easily verified in almost all known concrete examples of interpolation pairs. We give the details of the verification in the case E_0, E_1 are L^p-spaces over an arbitrary locally compact space X with respect to a positive measure on X. Lions [3] has verified the condition in another important case.

Interpolation spaces

A pair E_0, E_1 of Banach spaces is called an interpolation pair, if E_0 and E_1 are continuously embedded in some separated topological linear space $\mathcal{E}$. One verifies easily that $E_0 \cap E_1$ and $E_0 + E_1$ are Banach spaces in the norms