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Studies on a convolution inequality

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Introduction

Let μ be a positive, regular measure with total mass one on a locally compact Abelian group G (we refer to Appendix E 1 in Rudin [12] for the definition of regular measure). For certain classes of regular measures ν the operation $\nu * \mu$ can be defined and gives a new regular measure. We consider classes such that, if ν is in a certain class, then the same is true for (cf. Rudin [12] 1.3.4) $k * \nu$, where k is any continuous function with compact support, and

$$(k * \nu) * \mu = k * (\nu * \mu). \tag{0.1}$$

The measure $\nu * \mu$ can be interpreted as a "weighted mean value" of ν. The starting point of this paper is the following problem:

Let $\{\nu\}$ be a given class and consider the inequality

$$\nu - \nu * \mu \geqslant 0. \tag{0.2}$$

Which are the solutions in the given class and what properties do they have?

Suppose ν is such a solution. Let k be an arbitrary non-negative continuous function with compact support and form $\varphi = k * \nu$. It easily follows from (0.1) and (0.2) that

$$\varphi - \varphi * \mu \geqslant 0. \tag{0.3}$$

Obviously ν can be completely described by varying the function k. Hence the solution of the original problem can be characterized using the continuous solutions of (0.3). This gives a reason for our choice to confine the investigations of this paper to classes of continuous solutions of (0.3). At some instances in the forthcoming discussions, however, we shall mention the implications on the original problem.

In § 2, we let μ be arbitrary and study a class $\{\varphi\}$ of continuous functions which satisfy (0.3) and which are bounded from below. In Theorem 2.2, conditions are given which are necessary and sufficient for the existence of such solutions of (0.3), non-trivial in the sense that the strict inequality holds in a set of positive Haar measure. An equivalent criterion is given in Lemma 2.1, namely that for some neighborhood $\hat{O}$ of zero in the dual group $\hat{G}$ of G there exists a constant C such that

$$\int_{\hat{O}} \operatorname{Re}\left\{\frac{1}{1+\varepsilon-\hat{\mu}(\hat{x})}\right\} d\hat{x} \leqslant C \quad \text{for all} \quad \varepsilon > 0. \tag{0.4}$$

In special cases (0.4) has been considered by many authors, cf. e.g. Chung and Fuchs [4]. We find that in this case we always have non-trivial solutions which are bounded.