ARKIV FÖR MATEMATIK Band 7 nr 11

1.66 13 2 Communicated 26 October 1966 by T. NAGELL and O. FROSTMAN

High submodules and purity

By BO T. STENSTRÖM

1. Introduction

The N-high subgroups of an abelian group G were defined by Irwin [5] as maximal subgroups having zero intersection with the given subgroup N of G. In this note we extend some well-known relations between neat and N-high subgroups ([2], § 28 and [4], p. 327) to abelian categories and in particular to modules over general rings. As an application we will generalize a characterization of intersections of neat subgroups, due to Rangaswamy [7]. The term "high" will here be used in a sense more general than that it has in [5].

Notation. $\mathcal{A}$ is an abelian category in which every object M has an injective envelope $E(M)$. For any subobject L of M we consider $E(L)$ as a well-defined subobject of $E(M)$.

2. High subobjects

Let M be an object in $\mathcal{A}$ with a given subobject K. A subobject L of M is called *K-high* if $L \cap K = 0$ and L is maximal with respect to this. K-high subobjects do exist for any K ([3], p. 360). We obviously have

Proposition 1. *A subobject L of M is K-high if and only if the composed morphism $K \to M \to M/L$ is an essential monomorphism.*

Corollary. *If L is K-high in M, then*

(i) *$L+K$ is essential in M.*
(ii) $E(M)=E(L) \oplus E(K)$.

The K-high subobjects of M may be described in terms of injective envelopes, as was done in [5] and [6] for abelian torsion groups.

Proposition 2. *The K-high subobjects of M are just the intersections of M with complementary summands of $E(K)$ in $E(M)$.*

Proof. If L is K-high, then $E(M)=E(L) \oplus E(K)$ by the corollary, and $L=E(L) \cap M$ since also $E(L) \cap M \cap K=0$. Conversely, suppose $E(M)=E(K) \oplus H$. Then $H \cap M \cap K=0$, and if L is K-high in M with $L \supset H \cap M$, then $E(L) \supset E(H \cap M)=H$. Clearly it follows that $E(L)=H$, and $H \cap M=E(L) \cap M=L$ is K-high.