Communicated 23 April 1952 by MARCEL RIESZ and ARNE BEURLING

Isomorphisms of abelian group algebras

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Let G and H be locally compact groups with group algebras $L(G)$ and $L(H)$ respectively. If G and H are isomorphic groups, the correspondence between points of G and H given by the isomorphism induces an isomorphism of the group algebras. The purpose of this paper is to make a contribution to the converse question: assuming that $L(G)$ and $L(H)$ are isomorphic algebras, under what conditions can it be asserted that the underlying groups are isomorphic?

As a case of the problem, one can ask when an automorphism of a single group algebra is induced by an automorphism of the underlying group. A discussion appeared in the author's thesis (Harvard, 1950)[1], where the group was assumed to be commutative. The principal result there stated was Theorem 3 of this paper, except that only automorphisms were considered. Further information is hard to obtain even for simple groups; for example, it is not known whether there are any automorphisms of the algebra on the line, except a few obvious and trivial ones.

At a late stage of this work I learned that J. WENDEL has obtained Theorem 3 (for an operator assumed to be isometric) even for non-abelian group algebras. More recently he has established all of Theorem 3. While his methods and mine undoubtedly are related, it is difficult to make direct comparisons because of the complexity of the general case. I am grateful to Dr. WENDEL for correspondence about the problem, and for a summary of [7] before it appeared in print.

We shall consider only abelian groups, where the Fourier transform is a convenient tool. Our main result, Theorem 4, asserts the following: If T is an operator mapping $L(G)$ isomorphically onto $L(H)$ *with bound less than two*, where G and H are locally compact abelian groups, and if the dual group of G or of H is connected, then G and H are isomorphic groups, and T is the natural isomorphism of algebras induced by the group isomorphism. Since Theorem 3 is a corollary of the more complicated methods used here, we are not giving its original proof.

The argument used to prove Theorem 4 is a modification of a proof of A. BEURLING for the following theorem: if for each real λ

$$e^{i\lambda\varphi(x)}$$

[1] Most of the content of §§ 1 and 2 and Theorem 3 appeared in my thesis. I am pleased to record my indebtedness to Professor L. H. LOOMIS, who directed the thesis. The other theorems are generalizations of unpublished results of Professor A. BEURLING, to whom I am obliged for much advice, and for permission to publish these theorems depending essentially on his work.