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On distribution functions with a limiting stable distribution function

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1. Introduction[1]

The general stable d.f.'s[2] have been introduced by P. Lévy[3] who defined them implicitly by help of their characteristic functions and explicitly as limiting distribution functions. To every α, $0 < \alpha \leq 2$ there belong stable distribution functions $G_\alpha(x)$ and these have the following property.[4] If $\ast$ denotes the convolution and σ_1, σ_2 and σ are positive numbers with

$$\sigma_1^\alpha + \sigma_2^\alpha = \sigma^\alpha \tag{1}$$

we have

$$G_\alpha\left(\frac{x}{\sigma_1}\right) \ast G_\alpha\left(\frac{x}{\sigma_2}\right) = G_\alpha\left(\frac{x}{\sigma}\right). \tag{2}$$

In the following we shall only use the property (2) and the fact that $G_\alpha(x)$ has derivatives of bounded variation of all orders.[5]

Let now $F(x)$ denote a d.f. and let $F^{*n}(x)$ denote the n-fold convolution of $F(x)$ with itself. W. Doeblin has given necessary and sufficient conditions which $F(x)$ must satisfy, if $F^{*n}(b_n x)$ shall converge to a stable d.f. $G_\alpha(x)$, $0 < \alpha < 2$.[6] If $\alpha = 2$ then $G_\alpha(x)$ is the normal d.f. and the conditions for convergence are then well known.

Our method can be used to get the conditions for convergence and we shall return to this problem later. Here we shall give estimations of the remainder term

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[1] Mr. Kai Lai Chung drew my attention to the general stable d.f.'s in a discussion which I had with him on the application of my methods.

[2] d.f. — read distribution function(s).

[3] P. Lévy (1), pp. 94–97, 198–204.

[4] We call α the exponent of the stable d.f.

[5] We omit the singular case, when $G_\alpha(x)$ is discontinuous.

[6] W. Doeblin (1), pp. 71–96.