ARKIV FÖR MATEMATIK Band 2 nr 22

Communicated 26 November 1952 by F. CARLSON

A note on recurring series

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We wish to prove the following theorem, which extends to any field of characteristic 0 a result, proved by SKOLEM [4] in the field of rational numbers and by MAHLER [2] in the field of algebraic numbers.

Theorem. *In a field of characteristic* 0, *let a sequence*

$$c_\nu \qquad \nu = 0, 1, 2, \ldots$$

satisfy a recursion formula of the type

$$c_\nu = \alpha_1 c_{\nu-1} + \alpha_2 c_{\nu-2} + \cdots + \alpha_n c_{\nu-n} \qquad \nu = n, n+1, n+2, \ldots.$$

If $c_\nu = 0$ *for infinitely many values of* ν, *then those* c_ν *that are equal to zero occur periodically in the sequence from a certain index on.*

It will be shown by an example that the restriction for the characteristic is essential (section 6).

From the theorem can be deduced a characterization of those sequences $\{c_\nu\}$ that contain 0 (or: any number) an infinity of times (see MAHLER [2]). In particular, only a finite number of the c_ν can be equal to zero if the quotient of two different roots of the equation $1 = \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n$ is never a root of unity.

SKOLEM and MAHLER used for their proofs a p-adic method, due to SKOLEM [3]. Our proof will closely follow that of MAHLER, and is partly built on it.

1. A sequence $\{c_\nu\}$ in any field may be considered as the TAYLOR-coefficients of a rational function if it satisfies a linear recursion formula as above. By resolving this rational function into partial fractions it is possible to get an explicit expression for the c_ν. In a field of characteristic 0 we get

$$c_x = \sum_{j=1}^{m} A_j^x P_j(x) \qquad x = 0, 1, 2, \ldots,$$

where the $P_j(x)$ are polynomials whose coefficients, together with the A_j, are algebraic over the field that is generated by the c_ν. Therefore, to prove the theorem, it is sufficient to prove the following lemma.

Lemma. *Let the function* $F(x)$ *be defined by*

$$F(x) = \sum_{j=1}^{m} A_j^x P_j(x),$$