1.68 07 1 Communicated 24 April 1968 by Lennart Carleson

A problem of Newman on the eigenvalues of operators of convolution type

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In [1] Newman has studied the problem of uniqueness of the class of equations

$$\lambda F(x) - \int_E F(t)\, K(x-t)\, dt = G(x) \quad \text{for} \quad x \in E$$

under the restriction that K has compact support. However, the result in [1] is true also without that restriction:

Theorem 1. *Let G be a locally compact abelian group with Haar measure dt and let $K(x) \in L^1(G)$. Then for any measurable set E*

$$\lambda F(x) - \int_E F(t)\, K(x-t)\, dt = 0 \quad \text{for} \quad x \in E \quad \text{and} \quad F \in L^\infty \Rightarrow F \equiv 0$$

if $\lambda \notin H_k = \overline{CH\{\hat{K}(\xi) \mid \xi \in \hat{G}\}}$ = the closed convex hull of the values assumed by the Fourier transform $\hat{K}$ of K.

An equivalent theorem is obtained by looking at the class of operators on L^∞

$$\mathbf{K}_E F = \begin{cases} F \ast K & \text{for} \quad x \in E, \\ 0 & \text{for} \quad x \notin E, \end{cases}$$

where the kernel $K \in L^1$. The theorem then states that for any measurable set E, $\mathbf{K}_E$ has all its eigenvalues inside H_K. Thus H_K is a bound, uniform in E, for the eigenvalues of $\mathbf{K}_E$. The question of the "best" uniform bound has not been settled. The eigenvalue problem when $G = \mathbf{R}$ or $\mathbf{Z}$ has been solved in the cases $E = (-\infty, \infty)$, $(-\infty, 0)$ and $(0, \infty)$ (see e.g. Krein [2]). Together these eigenvalues form the set

$$A_K = \{\hat{K}(\xi) \mid \xi \in \hat{G}\} \cup \{\lambda \mid \operatorname{ind}(\lambda - \hat{K}) = (2\pi)^{-1} \int_{-\infty}^{\infty} d_\xi \arg(\lambda - \hat{K}(\xi)) \neq 0\},$$

i.e. the set of points on or "inside" the curve described by the Fourier transform $\hat{K}$. Consequently, if M_K is the best uniform bound for the eigenvalues then