ARKIV FÖR MATEMATIK Band 7 nr 39

1.68 02 1 Communicated 24 January 1965 by HARALD CRAMÉR and LARS GÅRDING

Convolutions of random functions

By HARALD BERGSTRÖM

1. Introduction

Let a probability space $[X, B, P]$ be given and denote the set of real numbers by R. Let L^p be the class of all random variables ξ with $E|\xi|^p < +\infty$, $p \geq 1$ and the norm $\|\xi\|_p = E^{1/p}|\xi|^p$. A random function is said to belong to L^p if $\xi(t) \in L^p$ for $t \in R$. We shall consider the topology in L^p given by this norm and deal with limits, continuity, etc., with respect to it. Then we talk about limits, continuity (L^p) or L^p-limits, L^p-continuity, etc.

A random function ξ is called a.s. non-decreasing if it is real and $\xi(t_1) \leq \xi(t_2)$ a.s. for any pair (t, t_2), $t_1 \leq t_2$. (Since we do not require that the random functions are separable, the sample functions need not be non-decreasing for a.s. all $x \in X$.) The L^p-limits $\xi(t+)$ and $\xi(t-)$ exist for such a random function (Theorem 2.1). If $\xi(t) = \frac{1}{2}[\xi(t-) + \xi(t+)]\,(L^p)$ we say that ξ is L^p-mean-continuous at that point and if such a relation holds for all t we say that ξ is L^p-mean-continuous. Let M^p be the class of L^p-mean-continuous a.s. non-negative, a.s. non-decreasing random functions and let $V^p = R(M^p)$ be the linear closure of M^p over R. We shall define a generalized convolution $\xi \circledast \eta \in V^p$ for $\xi \in V^{q_1}$, $\eta \in V^{q_2}$, $q_1 \geq 1$, $q_2 \geq 1$, $1/q_1 + 1/q_2 \geq 1/p$ and show that the commutative and associative laws hold for this convolution.

Let $M_0^p = \{\xi : \xi \in M^p,\ \xi(-\infty) = 0 \text{ a.s.}\}$ and let V_0^p be the linear closure of M_0^p. The L^p-FS-transform (F.S. read Fourier–Stieltjes) of $\xi \in V_0^p$ will be defined in section 5 as an RS-integral in respect to the L^p-norm and it will be shown that $\xi \circledast \eta$ has the L^p–FS-transform $\hat{\xi} \cdot \hat{\eta}$ when $\xi \in V_0^{q_1}$ and $\eta \in V_0^{q_2}$ have the L^{q_1}-FS-transform $\hat{\xi}$ and L^{q_2}-FS-transform $\hat{\eta}$ respectively $(1/q_1 + 1/q_2 \leq 1/p,\ q_1 \geq 1,\ q_2 \geq 1,\ p \geq 1)$. In a forthcoming paper [2] we shall prove a generalized Bochner theorem which gives necessary and sufficient conditions for a random function to be the L^p-FS-transform of an a.s. non-decreasing random function belonging to V_0^p. Then it is also possible to define the convolution of random functions with the help of L^p-RS-transforms in such a way that the two definitions agree. We have also given limit theorems for convolution products of random functions [3].

In many cases the generalizations of theorems for functions on the real line to corresponding theorems for random functions are quite simple and we can refer to [1] for details in the proofs.

2. The linear space V^p

The following simple lemma will frequently be used.

Lemma 2.1. *If ξ and η are* a.s. *non-negative random variables belonging to L^p and if $\xi \geq \eta$* a.s. *then*