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Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$ III

By GUNNAR ARONSSON

The present paper is a continuation of the papers [1] and [2]. These papers treat the problem of minimizing the functional

$$H(f) = \sup_x F(x, f(x), f'(x))$$

over the class $\mathfrak{J}$ of all absolutely continuous functions $f(x)$ which satisfy the boundary conditions $f(x_1) = y_1$ and $f(x_2) = y_2$. The discussion in [1] and [2] is mainly concerned with the existence and the properties of absolutely minimizing functions (defined in [1], p. 45) and unique minimizing functions. The question of the existence of a minimizing function is also treated in [2] and it is shown by an example ([2], p. 429) that a minimizing function *in general* need not have any of the properties proved for a.s. minimals ([2], Theorem 9′). However, if $F(x, f(x), \omega(x, f(x))) < M_0$ holds for a minimizing function $f(x)$, then $f(x)$ is a unique minimizing function (and hence $f(x)$ is smooth and $F(x, f(x), f'(x)) = M_0$). This is proved below and a few immediate consequences of this theorem are also discussed.

We assume that $F(x, y, z)$ satisfies the following conditions:

1. $F(x, y, z) \in C^1$ for $x_1 \leqslant x \leqslant x_2$ and all y, z.
2. There is a continuous function $\omega(x, y)$ such that

$$\frac{\partial F(x, y, z)}{\partial z} \text{ is } \begin{cases} >0 & \text{if } z > \omega(x, y), \\ =0 & \text{if } z = \omega(x, y), \\ <0 & \text{if } z < \omega(x, y). \end{cases}$$

3. $\lim_{|z| \to \infty} F(x, y, z) = +\infty$ if x and y are fixed.

A function $f(x)$ is admissible (belongs to $\mathfrak{J}$) if and only if $f(x)$ is absolutely continuous on $[x_1, x_2]$ and satisfies $f(x_1) = y_1$, $f(x_2) = y_2$. Put $M_0 = \inf_{f \in \mathfrak{J}} H(f)$. Thus, a function $f_0(x) \in \mathfrak{J}$ is a minimizing function if and only if $H(f_0) = M_0$.

Theorem. *Assume that $f(x)$ is a minimizing function such that*

$$F(x, f(x), \omega(x, f(x))) < M_0 \quad \textit{for} \quad x_1 \leqslant x \leqslant x_2.$$

Then $f(x)$ is the only minimizing function. Furthermore, $f(x) \in C^2[x_1, x_2]$ and $F(x, f(x), f'(x)) = M_0$ for $x_1 \leqslant x \leqslant x_2$. (Compare Theorem 6′ in [2].)