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Integral representation of certain linear functionals

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§ 1. Introduction

Any addition to the already enormous literature on integral representations for abstract linear functionals on general function spaces must show good cause for its appearance. The justification submitted for the present paper consists of three assertions: (1) the exhibition of integral representations is carried out here under what must be truly minimal hypotheses; (2) the limitations inherent in any possible integral representation are clearly indicated by means of examples; (3) the results obtained have applications in the study of certain topological algebras associated with groups.

We are concerned in the present communication with the following problem. Let there be given a set X, a linear space $\mathfrak{F}$ of real or complex functions defined on X, and a linear functional I defined on $\mathfrak{F}$. Under what conditions is it possible to find a finitely (or countably) additive measure γ^*, defined on a certain family of subsets of X, such that

1.1 $$I(f) = \int_X f(x)\, d\gamma^*(x)$$

for all, or at least part of, the functions f in $\mathfrak{F}$? A reasonably satisfactory answer to this question is contained in the present paper.

We use the following symbols and terminology. For a set X and a family of subsets $\mathcal{A}$ of X, the symbol $\mathcal{R}(\mathcal{A})$ denotes the smallest ring of sets containing $\mathcal{A}$ (i.e., the smallest family of sets containing $\mathcal{A}$ and closed under the formation of finite unions and differences). The symbol $\mathcal{S}(\mathcal{A})$ denotes the smallest σ-ring containing $\mathcal{A}$ (i.e., the smallest ring containing $\mathcal{A}$ which is closed under the formation of countable unions). The symbol $\mathcal{H}(\mathcal{A})$ denotes the family of all subsets Q of X such that for some $A \varepsilon \mathcal{A}$, $Q \subset A$. For $P \subset X$, the symbol χ_P denotes the characteristic function of P, i.e., the function equal to 1 on P and 0 on $X \cap P'$. A function φ defined on a ring $\mathcal{B}$ of subsets of X such that $0 \leqq \varphi \leqq +\infty$ is said to be a finitely additive measure on $\mathcal{B}$ if φ is not identically $+\infty$ and if the relation $\varphi(A \cup B) = \varphi(A) + \varphi(B)$ holds for all A, $B \varepsilon \mathcal{B}$ such that $A \cap B = 0$. A finitely additive measure on $\mathcal{B}$ is said to be countably additive if $\{A_n\}_{n=1}^{\infty} \subset \mathcal{B}$, $\bigcup_{n=1}^{\infty} A_n \varepsilon \mathcal{B}$, and $A_n \cap A_m = 0$ for $n \neq m$ imply that $\varphi(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \varphi(A_n)$. If φ is permitted to assume values in the closed interval $[-\infty, 0]$, then certain complications in the definition enter. We shall have occasion to consider negative measures only in 4.8 and 4.9, and in