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On an unsolved question concerning the Diophantine equation $Ax^3 + By^3 = C$

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§ 1.

The Diophantine equation

$$x^3 + Dy^3 = 1 \tag{1}$$

was solved completely by B. Delaunay [1][1] who showed that it has at most one solution in integers x and y when $y \neq 0$; if x, y is an integral solution, then

$$\eta = x + y\sqrt[3]{D} \tag{2}$$

is the fundamental unit of the ring $\boldsymbol{R}(1, \sqrt[3]{D}, (\sqrt[3]{D})^2)$.

T. Nagell [2], [3], [4], and [5] proved the same theorem independently of Delaunay and, moreover, a stronger form of the latter part of the theorem.

Nagell [4] and [5] proved that η is the fundamental unit of the field $\boldsymbol{K}(\sqrt[3]{D})$, except when $D = 19$, 20, and 28, in which cases η is the square of the fundamental unit. These values of D correspond to the solutions $x = -8$, $y = 3$; $x = -19$, $y = 7$; and $x = -3$, $y = 1$.

To solve (1), one has thus to determine the fundamental unit of $\boldsymbol{K}(\sqrt[3]{D})$, and to examine whether it has the form (2) or not.

Nagell [4] generalized these results and showed that the Diophantine equation

$$Ax^3 + By^3 = C, \tag{3}$$

where $C = 1$, or $C = 3$, where A and B are > 1 when $C = 1$ and where AB is not divisible by 3 when $C = 3$, has at most one solution in integers x and y.

He also established the following result: Put $A = ac^2$ and $B = bd^2$, where a, b, c, and d are positive integers, relatively prime in pairs, and possessing no square factors. Then, if x, y is a solution, one has

$$\eta = \frac{1}{C}(x\sqrt[3]{A} + y\sqrt[3]{B})^3 = \xi^{2^r}, \tag{4}$$

[1] Figures in [ ] refer to the Bibliography at the end of this paper.