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On equivalent analytic functions

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With 1 figure in the text

1. We denote by $\mathcal{R}$ the class of functions $f(z)$ that are analytic in a circle $|z| \leq R$. Two functions $f(z)$ and $g(z)$ of $\mathcal{R}$ are called *equivalent* if $f(z)$ is transformed into $g(z)$ by

(i) multiplication with a constant of modulus 1,
(ii) a transformation $z' = z e^{i\alpha}$ (α real),
(iii) replacing of all coefficients in the power series of $f(z)$ by their conjugate values.

Thus

$$g(z) = e^{i\beta} f(z e^{i\alpha}) \quad \text{or} \quad g(z) = e^{i\beta} \overline{f(\bar{z} e^{i\alpha})}.$$

We also call two harmonic functions $u(z)$ and $u_1(z)$ or two curves c and c_1 equivalent if one is transformed into the other by

(i) rotating the z-plane an angle α about $z = 0$,
(ii) reflection in a straight line through $z = 0$.

Thus

$$u_1(z) = u(z e^{i\alpha}) \quad \text{or} \quad u_1(z) = u(\bar{z} e^{i\alpha}).$$

We obtain immediately that if $f(z)$ and $g(z)$ of $\mathcal{R}$ are equivalent, then the harmonic functions $\log |f|$ and $\log |g|$ are equivalent.

Let $f(z)$ belong to $\mathcal{R}$. Given $r \leq R$, we put $z = r e^{i\varphi}$ and define $e_f(r, a)$ as the set of φ, $0 \leq \varphi \leq 2\pi$, such that $|f(r e^{i\varphi})| \leq a$ in e_f. Denoting by $\Phi_f(r, a)$ the measure of e_f we will call Φ_f the M-function of $f(z)$.

According to the definition, Φ_f is a non-decreasing function of a. If $M(r)$ and $m(r)$ denote as usual the maximum and minimum of $|f(z)|$ for $|z| = r$, then $\Phi_f = 0$ for $a < m(r)$ and $\Phi_f = 2\pi$ for $a > M(r)$. It is easily seen that if $f(z)$ and $g(z)$ are equivalent, then Φ_f and Φ_g are identical for all $r \leq R$.

In the following we always exclude the case that $f(z)$ is a power of z, $f(z) = a z^m$. In this case the obtained results are trivial. Therefore we assume that $m(r) < M(r)$,[1] and that $\Phi_f(r, a)$ is increasing in the interval $m(r) \leq a \leq M(r)$.

[1] There is at most one value of r for which $m(r) = M(r)$. This special value is of no interest here. See Blumenthal (1), Valiron (2).