Integration of Fokker-Planck's equation in a compact Riemannian space

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1. The result. Let $R$ be an $n$-dimensional compact Riemannian space with the metric $ds^2 = g_{ij}(x)dx^i dx^j$, and consider a temporally homogeneous Markoff process on $R$ for which $P(t, x, y)$, $t > 0$, is the transition probability that the point $x$ be transferred to $y$ after the elapse of $t$ units of time. We assume that $P(t, x, y)$ is continuous in $(t, x, y)$ and hence satisfies Smoluchovski's equation

$$P(t+s, x, y) = \int_R P(t, x, z)P(s, z, y)dz \quad (t, s > 0),$$

where the volume measure

$$dz = Vg(z)dz^1 \ldots dz^n, \quad g(z) = \det [g_{ij}(z)],$$

and the probability hypothesis

$$P(t, x, y) \geq 0, \quad \int_R P(t, x, y)dy = 1.$$

The "continuity" of the transition process $P(t, x, y)$ may be defined as follows.¹ Let $L^1(R)$ be the Banach space of functions $f(x)$ integrable with respect to $dx$ over $R$. There exist functions $f(x)$ dense in $L^1(R)$ for which the so-called Fokker-Planck equation holds:

$$\frac{\partial}{\partial t}f(t, x) = A \cdot f(t, x) \quad (t \geq 0),$$

$$f(0, x) = f(x),$$

where the operator $A$ is defined by