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Integration of Fokker-Planck's equation in a compact Riemannian space

By KÔSAKU YOSIDA

1. The result. Let R be an n-dimensional compact Riemannian space with the metric $ds^2 = g_{ij}(x)\, dx^i dx^j$, and consider a *temporally homogeneous* MARKOFF *process* on R for which $P(t, x, y)$, $t > 0$, is the *transition probability* that the point x be transferred to y after the elapse of t units of time. We assume that $P(t, x, y)$ is continuous in (t, x, y) and hence satisfies SMOLUCHOVSKI's equation

$$(1.1) \qquad P(t+s, x, y) = \int_R P(t, x, z)\, P(s, z, y)\, dz \quad (t, s > 0),$$

where the volume measure

$$dz = \sqrt{g(z)}\, dz^1 \ldots dz^n, \quad g(z) = \det\,[g_{ij}(z)],$$

and the *probability hypothesis*

$$(1.2) \qquad P(t, x, y) \geqq 0, \quad \int_R P(t, x, y)\, dy = 1.$$

The "continuity" of the transition process $P(t, x, y)$ may be defined as follows.[1] Let $L^1(R)$ be the Banach space of functions $f(x)$ integrable with respect to dx over R. There exist functions $f(x)$ dense in $L^1(R)$ for which the so called FOKKER-PLANCK *equation* holds:

$$(1.3) \qquad \frac{\partial}{\partial t} f(t, x) = A \cdot f(t, x)\ (t \geqq 0), \qquad \begin{aligned} f(t, x) &= \int_R f(y)\, P(t, y, x)\, dy \quad (t > 0), \\ f(0, x) &= f(x), \end{aligned}$$

where the operator A is defined by

[1] A. KOLMOGOROFF: Zur Theorie der stetigen zufälligen Prozesse, Math. Ann. 108 (1933). W. FELLER: Zur Theorie der stochastischen Prozesse, Math. Ann., 113 (1937).