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Translation invariant convex metrics

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1. Introduction

In this paper we shall consider metric linear topological spaces with a metric that is translation invariant and convex. A metric d on a set of points M will be called convex if for all points $x, y \in M$ and for all $\delta_1 \geqslant 0$, $\delta_2 \geqslant 0$ such that $\delta_1 + \delta_2 = d(x, y)$ there exists a point $z \in M$ with $d(x, z) = \delta_1$ and $d(z, y) = \delta_2$.

The concept of a convex metric was first introduced by Menger [3]. According to his definition a metric space M is convex if for all points $x, y \in M$ there exists a metric midpoint (i.e. $\delta_1 = \delta_2$ above).

Only slight modifications are needed to make our proofs valid under the weaker assumption of midpoint convexity. If the space is complete it follows from the theorems of Menger that the definitions are equivalent.

It can be proved that an invariant convex metric defining the topology of a linear topological space is a norm if and only if the spheres around the origin are convex. On the other hand such a space might have spheres which are not convex even if it is locally convex. Rådström [5] has given an example of this in $L^1\,[0, 1]$. However, assuming local convexity we shall show that the space is isomorphic to a normed space. We also give an explicit expression of an equivalent norm in the convex metric. Furthermore we shall show that if the space is reflexive and separable it is isometric to a Banach space.

2. Auxiliary theorems on convex metrics

If A and B are subsets of a linear space X we write $A + B = \{x \in X;\ x = a + b,\ a \in A,\ b \in B\}$. A family of subsets of X will form a commutative semigroup if it is closed under this addition. Such a semigroup will be called a one-parameter semigroup if there is an application $\delta \to A(\delta)$ from the positive real numbers onto the semigroup satisfying

$$A(\delta_1 + \delta_2) = A(\delta_1) + A(\delta_2) \tag{1}$$

The definition of a one-parameter semigroup usually contains some assumption of continuity, but this will not be needed here and is omitted.

Let X be a metric linear topological space with a metric d that is translation invariant. Denote the open sphere of radius δ around the origin by $S(\delta)$. It then follows (see [4]) that the metric is convex if and only if the sets $\overline{S(\delta)}$ form a one-parameter semigroup, the bar denoting the closure. We need the "only if" part of this statement for the open spheres.