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Pseudo-lattices: Theory and applications

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The notion of a partially ordered set is well-known. It is also known that a quasi-ordered (pre-ordered) set is a system consisting of a set X and a binary relation $\geqslant$ satisfying the following laws:

P_1: For all x in X, $x \geqslant x$ (Reflexive); P_2: If $x \geqslant y$ and $y \geqslant z$, then $x \geqslant z$ (Transitive).

In a quasi-ordered set if a least upper bound or a greatest lower bound of some subset exists it may not exist uniquely, since we do not necessarily have antisymmetry for the quasi-ordering. This motivates the following:

Definition 1. *A quasi-ordered set is called a pseudo-lattice iff any two elements have at least one least upper bound and at least one greatest lower bound.*

Before we construct new pseudo-lattices from given ones, we need more definitions:

Definition 2. *Let $\geqslant$ and $\gg$ be two quasi-orderings on a given set X, then $\gg$ is stronger than $\geqslant$ iff $x \geqslant y$ implies $x \gg y$.*

Definition 3. *Let $(X, \geqslant)$ and $(Y, \geqslant)$ be two quasi-ordered sets, $f: X \to Y$ a mapping. f is order-preserving iff $a \geqslant b$ implies $f(a) \geqslant f(b)$. f is called bi-order-preserving iff*

(1) *$a \geqslant b$ implies $f(a) \geqslant f(b)$ and*
(2) *$f(a) \geqslant f(b)$ implies $a \geqslant b$.*

Definition 4. *Two quasi-ordered sets $(X, \geqslant)$ and $(Y, \geqslant)$ are called isomorphic iff there exists a bijective bi-order-preserving mapping f of X onto Y, i.e., iff there exists a one-to-one-mapping f of X onto Y such that $f(a) \geqslant f(b)$ iff $a \geqslant b$.*

Theorem 1. *Let X be a set, $(Y, \gg)$ a quasi-ordered set and $f: X \to Y$ a mapping. Then there exists a strongest quasi ordering $\geqslant_f$ on X under which f preserves ordering. Furthermore, $(X, \geqslant_f)$ is a pseudo-lattice if $(Y, \gg)$ is a pseudo-lattice and f an onto mapping.*

Proof A binary relation $\geqslant_f$ on X is defined by setting $a \geqslant_f b$ iff $f(a) \gg f(b)$. Evidently $\geqslant_f$ is a quasi-ordering on X under which f preserves ordering. Suppose f preserves ordering under a quasi-ordering $\geqslant$ on X. Then $a \geqslant b$ implies $f(a) \gg f(b)$. This in turn implies $a \geqslant_f b$. Thus $\geqslant_f$ is the strongest quasi-ordering on X under which f preserves ordering.