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The optimal number of faces in cubical complexes

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1. Introduction

Let a *cubical complex* C be a set of faces of an n-dimensional cube, such that if a face of dimension r, $1 \leqslant r \leqslant n$, belongs to C, then all lower dimensional faces of this r-face belong to C. J. B. Kruskal suggested in [7] the problem to optimize the number of s-dimensional faces for complexes which contain a fixed number of faces of dimension r. What is required is to determine the maximum possible number of s-faces if $r<s$ and the minimum possible number of s-faces if $r>s$ and the minimum possible number of s-faces if $r>s$ that C can have if the number of r-faces in C is given. In the special case $r=0$, $s=1$ this optimization problem has been solved by L. H. Harper and A. J. Bernstein in [4] and [1], and also by J. H. Lindsey II in [8].

For simplicial complexes a similar optimization problem was solved in full generality by J. B. Kruskal in [6]. G. Katona has also solved this problem in [5] not aware of Kruskal's solution. Another different solution can be obtained by the method used by G. F. Clements and B. Lindström in [2]. By a similar method I will be able to solve Kruskal's problem for cubical complexes for any r and s.

We shall also consider the problem to maximize a non-decreasing function of the dimensions of faces in a cubical complex and apply the result to a determinant defined by means of the Möbius function of the complex.

For the convenience of the reader we shall now give an outline of the method to be used in this paper.

A major step towards the solution of the problem is to find a suitable total ordering of all faces in the n-cube. Then we define the replacement operator R. If S_r is any set of cubical r-faces let RS_r be the $|S_r|$ first r-faces in the total ordering of faces ($|X|$ is the cardinality of the set X). We define the boundary operator ∂ such that ∂S_r is the set of all $(r-1)$-faces of elements in S_r.

The following inclusion is now crucial

$$\partial RS_r \subseteq R\partial S_r.$$

We shall prove this inclusion by induction over n, the dimension of the cube which contains the set S_r. To be able to use the induction hypothesis we have to introduce restricted replacement operators R_v, which operate in $n-1$ dimensions keeping the vth coordinate fixed. We first apply R_1 to S_r, then R_2 to $R_1 S_r$ etc. After applying R_n we apply R_1 and so on. It will turn out that the sets "converge" and we obtain a set T_r such that $R_v T_r = T_r$ for $v=1, \ldots, n$. In general is T_r distinct from RS_r. Therefore we have