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# Hardy-fields

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## Introduction

In his work [1] Hardy discusses the problem of describing how rapidly a function $f: R_1 \to R_1$, where $R_1$ is the system of real numbers, tends to infinity. This is a special case of the more general problem to describe how such a function behaves in the vicinity of $+\infty$. For simplicity we write $\infty$ instead of $+\infty$ in the sequel. We will only be interested in functions that are infinitely differentiable in a neighborbood of $\infty$. Thus let

$C^\infty = \{f \mid f$ is a real-valued function such that its domain lies in $R_1$ and contains a neighborhood of $\infty$ in which $f$ is infinitely differentiable$\}$.

To discuss the above problem we are naturally led to considering two functions $f, g$ belonging to $C^\infty$ as essentially equal, written $f \sim g$, if they are equal in a neighborhood of $\infty$, i.e. if there is a number $N$ such that $f(x) = g(x)$ for $x \geqslant N$. It is obvious that $\sim$ is an equivalence relation and that the equivalence classes are the same as the residue classes of the ideal $I = \{f \mid f \sim 0\}$ in the ring $C^\infty$. Let $R = C^\infty / I$ and let $I(f)$ denote the residue class of $f$ with respect to $I$. Then $R$ is a ring with differentiation where $(I(f))' = I(f')$ and it can be said to represent all the ways a function in $C^\infty$ can behave in a neighborhood of $\infty$.

The concept, Hardy-field, which will be studied in this paper, was essentially introduced in [1] but was, as far as I know, first formally defined in [2]. The definition given in [2] is equivalent to the following.

**Definition.** *A field $H$ contained in $R$, such that $y \in H$ implies that $y' \in H$, is said to be a Hardy-field.*

An example of a subset of $R$, which constitutes a Hardy-field, is the set of residue classes of the rational functions.

An intersection of an arbitrary non-empty family of Hardy-fields is a Hardy-field. Thus if $A \subset R$ and if there is a Hardy-field containing both a given Hardy-field $H$ and the set $A$ then there is also a smallest Hardy-field, denoted by $H\{A\}$, containing both $H$ and $A$. If $H\{A\}$ exists then $A$ is said to be adjoinable to $H$. If $A = \{a_1, ..., a_n\}$ then we write $H\{A\} = H\{a_1, ..., a_n\}$ instead of $H\{\{a_1, ..., a_n\}\}$ and we say that $a_1, ..., a_n$ are adjoinable to $H$.

It will be one of the main concerns in this work to extend a given Hardy-field in such a way that the extension has nice algebraic and analytic properties. A natural