ARKIV FÖR MATEMATIK Band 8 nr 19

1.69 02 1 Communicated* 22 January 1969 by ÅKE PLEIJEL and LENNART CARLESON

Subharmonic functions in a circle

By ULF HELLSTEN, BO KJELLBERG and FOLKE NORSTAD

1. Introduction

Let $u(z)$ be a subharmonic function of a complex variable z, defined in a circular region $|z| < R$. Let

$$m(r) = \inf_{|z|=r} u(z), \quad M(r) = \max_{|z|=r} u(z), \quad M(R) = \sup_{|z|<R} u(z).$$

A condition of the type
$$m(r) \leqslant \cos \pi\lambda M(r), \tag{1}$$

where λ is a number in the interval $0<\lambda<1$, has been found to give consequences concerning the variation of $M(r)/r^\lambda$. If $u(z)$ is subharmonic in the entire plane and if (1) holds for all $r>0$, then $M(r)/r^\lambda$ has a positive limit when $r\to\infty$ (see [1, 2, 4, 6]). An essential part of the proof of this is to show that, with a given value of $M(R)/R^\lambda$, the quotient $M(r)/r^\lambda$ must be bounded for $0<r<R$. We shall here make a closer study of this problem.

The special case $\lambda=\frac{1}{2}$ has long been known, this being the Milloux-Schmidt inequality (see, for example [5], p. 108–109):

$$M(r) \leqslant U_0(r), \quad \text{where } U_0(r) = \frac{4M(R)}{\pi} \arctan \sqrt{\frac{r}{R}}. \tag{2}$$

One consequence of (2) is that

$$\frac{M(r)}{\sqrt{r}} \leqslant \frac{4}{\pi} \frac{M(R)}{\sqrt{R}}. \tag{3}$$

In the general case $0<\lambda<1$, we prove the following.

Theorem

Suppose that $u(z)$ is subharmonic for $|z|<R$ and that $0<M(R)<\infty$. Let λ be a fixed number in the interval $0<\lambda<1$ and suppose that condition (1) is satisfied for $0<r<R$. Then there is an extremal subharmonic function,

$$U(z) = \operatorname{Re}\left\{\frac{2M(R)}{\pi} \tan \frac{\pi\lambda}{2} \int_0^{z/R} \frac{t^{\lambda-1}-t^{1-\lambda}}{1-t^2}\, dt\right\}, \quad |\arg z| \leqslant \pi, \tag{4}$$

for which (1) holds with equality and such that

$$M(r) \leqslant U(r). \tag{5}$$

* Manuscript partly rewritten, final shape 5 June 1969.