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Necessary and sufficient conditions for the hyperbolicity of polynomials with hyperbolic principal part

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0. Introduction

Let $P(\xi)=\sum_{|\alpha|\leqslant m} c_\alpha \xi^\alpha$ be a complex polynomial of degree m in the complex variables $\xi=(\xi_1,\ldots,\xi_{d+1})$, and let $P_m(\xi)=\sum_{|\alpha|=m} c_\alpha\xi^\alpha$ be its principal part. Let $(x_1,\ldots, x_{d+1})$ be real variables, and put $D_k=\partial/i\partial x_k$. A distribution $E(x)$ on R^{d+1} is said to be a fundamental solution of the differential operator $P(D)$ if $P(D)\,E(x)=\delta(x)$, the Dirac distribution. The operator $P(D)$ is said to be hyperbolic if it has a fundamental solution E with support in a proper cone K having its vertex at the origin (Gårding [5]). Let $N\in R^{d+1}$ be such that the halfspace $\langle x, N\rangle=x_1N_1+x_2N_2+\ldots+x_{d+1}N_{d+1}>0$ contains $\dot{K}=K-\{0\}$. Then

$$P_m(N)\neq 0,\ P(\xi+i\tau N)\neq 0 \quad \text{if} \quad \xi\in R^{d+1},\ \tau\in R,\ |\tau|>\tau_0 \tag{0.1}$$

for some τ_0. Conversely, this condition implies that $P(D)$ has a fundamental solution with support in some K such that $\langle x, N\rangle>0$ on $\dot{K}$ (Gårding [5], [4]).

When (0.1) holds, we say that P is hyperbolic with respect to N and denote by Hyp N the corresponding class of polynomials.

It follows that P_m is in Hyp N if P is, and that a homogeneous hyperbolic polynomial has only real characteristics. We shall, conversely, consider the problem of characterizing the lower order terms one may add to a homogeneous hyperbolic polynomial without loss of the hyperbolicity. In the case $d=1$, this problem has been solved completely by A. Lax [8]. A generalization of A. Lax's condition was given by Hörmander in [6]. His generalized condition is necessary but not sufficient when $d>1$.

A sufficient condition by Gårding [4] for a polynomial P to belong to Hyp N, if its principal part P_m does, is that the roots σ of $P(\sigma(\tau N+i\xi))=0$ tend to zero, uniformly in $\xi\in R^{d+1}$, when $\tau\to+\infty$. Gårding conjectured that this condition would be necessary too. (See footnote, page 50 in Gårding [4].)

In section 1 of this paper we shall prove Gårding's conjecture. We use a sufficient condition by Hörmander [6], which can be shown to be equivalent to that of Gårding, namely that P is weaker than P_m, i.e. that for some constant C we have

$$|P(\xi)|\leqslant C\tilde{P}_m(\xi),\ \xi\in R^{d+1}.$$

Here, when Q is a polynomial, we put

$$\tilde{Q}(\xi)=\Big(\sum_\alpha|\partial^\alpha Q(\xi)|^2\Big)^{\frac{1}{2}},\quad \partial=(\partial/\partial\xi_1,\ldots,\partial/\partial\xi_{d+1}).$$