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Compact groups and Dirichlet series

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1. In a previous paper [4] I have tried to show that the function-theoretic properties of Dirichlet series are associated with subalgebras of $L^1(-\infty, \infty)$, rather than with almost-periodic functions and compactifications of the line. Now I want to weaken that point by proving a convergence theorem for Dirichlet series, considered as Fourier series on a compact group, that does not apply to Fourier series of analytic type in general.

This paper carries on ideas introduced in [3], but does not refer to the theorems about cocycles proved there.

We start with a subgroup Γ of R_d, the discrete real line. The dual of Γ is a compact group K with normalized Haar measure σ. (The case where K is a circle is uninteresting, and is excluded.) A summable function f on K has Fourier series

$$f(x) \sim \sum_{\lambda} a(\lambda)\, \chi_\lambda(x) \quad (\lambda \in \Gamma), \tag{1}$$

where χ_λ is the character on K defined by $\chi_\lambda(x) = x(\lambda)$. For $p \geqslant 1$, $H^p(K)$ is the subspace of $L^p(K)$ consisting of those functions f in whose Fourier series $a(\lambda) = 0$ for all $\lambda < 0$.

A *Dirichlet sequence* in Γ is a sequence of real numbers λ_n in Γ such that

$$0 \leqslant \lambda_1 < \lambda_2 < \ldots; \qquad \lambda_n \to \infty.$$

Let $H^p(K, \Lambda)$ be the space of all functions f in $H^p(K)$ in whose Fourier series $a(\lambda) = 0$ except for λ in the Dirichlet sequence Λ. A Dirichlet sequence satisfies the *condition of Bohr* if there are constants c, k such that

$$(\lambda_{n+1} - \lambda_n)^{-1} \leqslant k \exp(c\lambda_n) \quad (n = 1, 2, \ldots).$$

This condition prevents λ_n from increasing too slowly, or from bunching up too densely. It is satisfied in the case of ordinary Dirichlet series: $\lambda_n = \log n$.

2. Let e_t be the element of K defined by $e_t(\lambda) = e^{it\lambda} (\lambda \in \Gamma)$, for each real number t. Then $K_0 = \{e_t\}$ is a dense subgroup of K of measure 0. To a measurable function f on K we associate functions $f_x(t) = f(x + e_t)$ defined and measurable, for almost every x, on the line. If f is in $L^2(K)$, then for almost every x the restriction f_x is square-