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A nullstellensatz for ordered fields

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For an ordered field k, a *realzero* of an ideal P in the polynomial ring $k[X] = k[X_1, ..., X_n]$ in n variables is a zero in $\bar{k}^{(n)}$, where $\bar{k}$ is the realclosure of k, the *real-variety* $\mathcal{V}_R(P)$ is the set of all realzeros of P, and, as usual, $\mathcal{J}(G)$, for any subset G of $\bar{k}^{(n)}$ is the ideal of all members of $k[X]$ that vanish all over G. Our nullstellensatz asserts:

$$\mathcal{J}\mathcal{V}_R(P) = \sqrt[R]{P} = \textit{realradical of } P,$$

where $\sqrt[R]{P}$ is the set of all $f(X)$ such that for some exponent m, some *rational* functions $u_i(X)$ in $k(X)$, and positive $p_i \in k$

$$f(X)^m(1 + \Sigma\, p_i u_i(X)^2) \in P.$$

The proof, which uses Artin's solution of Hilbert's 17th problem, and which grew out of an attempt to find an easier solution to the problem, is straight-forward, inspired in large part by Lang's elegant formulation of various extension theorems, especially Theorem 5, p. 278 [2]. We give a new proof of this theorem, and a generalization to finitely generated formally real rings over k (Theorem 1).

Throughout, k will be an ordered field. For any ordered field K, $\bar{K}$ is its real closure.

A simple consequence of Artin's work (see Theorem 13 and Lemma 1 of Jacobson, Chapter VI [1]) is:

Artin's Theorem. *Let k be an ordered field, let $K = k(T) \equiv k(T_1, ..., T_n)$ be a pure transcendental ordered extension of k, with T_i algebraically independent. Let $f(Y) \in k[T][Y]$ have a root in $\bar{K}$, let $u_1, ..., u_m$ be a finite set of nonzero elements of $k[T]$. There exists a homomorphism σ over k from $k[T]$ to $\bar{k}$ satisfying*

(i) $\sigma(u_i) \neq 0$, $1 \leq i \leq m$.
(ii) *$f^\sigma(Y)$ has a root in $\bar{k}$.*

Lang's Theorem (Lang, Theorem 5, p. 278 [2]). *Let k be an ordered field, let $k \xrightarrow{\tau} R$ be an order-embedding of k into a realclosed field R. Let K be a field containing k and admitting an order extending the order of k. Then for every finite subset E of K there exists a homomorphism ψ: $k[E] \to R$ extending τ.*

Proof. Suppose the theorem is known for the case where τ is the inclusion map $k \subset \bar{k}$. For general τ, the algebraic closure $\overline{\tau k}$ in R is a real closure of τk and also of k, so by the uniqueness theorem for real closures there exists ψ: $\overline{\tau k} \cong \bar{k}$ such that ψ is order preserving and $\psi\tau$ is the inclusion $k \subset \bar{k}$. By supposition there exists σ: $k[E] \to \bar{k}$. Then $\psi^{-1}\sigma$: $k[E] \to R$ extends τ.