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# On a problem of Smirnov

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From the theorem that every separable metric space is isometric with a subset of $C(0, 1)$ and the theorem that all separable Banach spaces are homeomorphic it follows that every separable metric space is homeomorphic with a subset of $L_2(0, 1)$. In this paper we shall construct a countable metric space which is not uniformly homeomorphic with any subset of $L_2(0, 1)$. This gives a negative answer to a question asked by Smirnov. This question is the theme in [3] where among other things it is proved that Euclidean $n$-space is uniformly homeomorphic with a bounded subset of $L_2(0, 1)$. The question is also treated in [2] where a result in the negative direction is obtained and in [1] where a stronger result is obtained.

## 1. A geometric property of $L_2(0,1)$

We shall say that a set of $2n+2$ points in a metric space is a double $n$-simplex if the points are written $a_1, a_2, \dots a_{n+1}, b_1, b_2, \dots b_{n+1}$. We shall call a pair of points $(a_i, a_j)$ or $(b_i, b_j)$ $i \neq j$ an edge and a pair of points $(a_i, b_k)$ a connecting line. We shall say that a metric space $M$ has generalised roundness $p$, if $p$ is the supremum of the $q$'s with the property: for every $n \geqslant 1$ and every double $n$-simplex in $M, \Sigma c_\alpha^q \geqslant \Sigma s_\beta^q$ where $c_\alpha$ runs through the lengths of all connecting lines and $s_\beta$ runs through the lengths of all edges. In [1] we defined roundness to be the supremum of the $q$'s for which the inequality holds for double 1-simplexes. It is obvious that the generalised roundness is not larger than the roundness. In [1] it was proved that $L_p(0, 1)$ $1 \leqslant p \leqslant 2$ has rundness $p$. If a metric space has the property that some pair of points $(a_1, a_2)$ has a metric middle point $m$ then its roundness and thus its generalised roundness is not larger than 2. We see this by choosing $b_1 = b_2 = m$.

Since in every double $(n-1)$-simplex there are $n^2$ connecting lines and $n(n-1)$ edges the generalised roundness of a metric space is $\geqslant 0$. If in a double $(n-1)$-simplex we put the lengths of all connecting lines $=\frac{1}{2}$ and the lengths of all edges $=1$ then it is easy to see that we get a metric space with generalised roundness $-{}^2\log(1-1/n)$ which tends to 0 as $n \to \infty$. If in a double $(n-1)$-simplex we put instead the lengths of all connecting lines $=(1-1/n)^{1/q}$, $(1-1/n)^{1/q} \geqslant \frac{1}{2}$, $q > 0$, it is easy to see that we get a metric space with generalised roundness $q$.

**Theorem 1.1.** $L_2(0, 1)$ *has generalised roundness* 2.

*Proof.* Since $L_2(0,1)$ has roundness 2, the generalised roundness is not larger than 2. Thus it is enough to prove $\Sigma c_\alpha^2 \geqslant \Sigma s_\beta^2$ for all double $n$-simplexes in $L_2(0, 1)$. This inequality is for a double $(n-1)$-simplex equivalent with the inequality