1.68 15 2 Communicated 27 November 1968 by OTTO FROSTMAN

On the nonexistence of uniform homeomorphisms between L_p-spaces

By PER ENFLO

The main result of this paper shows that an infinite-dimensional $L_{p_1}(\mu_1)$ is not uniformly homeomorphic with $L_{p_2}(\mu_2)$ if $p_1 \neq p_2$, $1 \leqslant p_i \leqslant 2$ (our conclusions will in fact be stronger). This gives an affirmative answer to a conjecture by Lindenstrauss [1]. The method used here is quite different from that suggested by Lindenstrauss. We will use the terminology of [1]. In the sequel we will consider $L_{p_1}(0, 1)$ but it is easy to see that with slight adjustments of the proofs the results hold for $L_{p_1}(\mu_1)$ and $L_{p_2}(\mu_2)$ as well.

1. A geometric property of L_p (0,1)

We shall say that a metric space has roundness p if p is the supremum of the set of q:s with the property: for every quadruple of points a_{00}, a_{01}, a_{11}, a_{10}

$$[d(a_{00}, a_{01})]^q + [d(a_{01}, a_{11})]^q + [d(a_{11}, a_{10})]^q + [d(a_{10}, a_{00})]^q \geqslant [d(a_{00}, a_{11})]^q + [d(a_{01}, a_{10})]^q \tag{1}$$

The triangle inequality shows that (1) is always satisfied if $q=1$. If the metric space has the property that every pair of points has a metric middle point, (1) is not satisfied for all quadruples if $q>2$. We see this by choosing a_{01} as the middle point between a_{00} and a_{11} and choosing $a_{01}=a_{10}$. Of course (1) is also satisfied for $q=p$.

Theorem 1.1. $L_p(0, 1)$, $1 \leqslant p \leqslant 2$, *has roundness* p.

Proof. We first prove that $\int_0^1(|f_{00}-f_{01}|^p + |f_{01}-f_{11}|^p + |f_{11}-f_{10}|^p + |f_{10}-f_{00}|^p - |f_{00}-f_{11}|^p - |f_{01}-f_{10}|^p)\,dt \geqslant 0$. We observe that it is enough to prove that the integrand is nonnegative. This is an inequality involving four real numbers and we can assume that the least of them is 0. Thus we have to prove $x^p + |z-x|^p + y^p + |z-y|^p - z^p - |y-x|^p \geqslant 0$. We observe that it is enough to prove this inequality for $1<p<2$. The inequality holds for $z=0$, $z=x$, $z=y$. The derivative with respect to z of the left side is not positive in the intervals $[0, \min(x, y)]$ and $[\min(x, y), \max(x, y)]$. Thus we can assume $0 \leqslant x \leqslant y \leqslant z$. We keep z fixed and observe that the inequality holds for $x=0$, $x=y$ and $y=z$. We then form the partial derivatives with respect to x and y of the left side and observe that both of them equals zero only when $x=y=z/2$. We finally observe that the inequality holds under this assumption. Thus the inequality is proved.