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On a conjecture of V. Bernstein

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1. Introduction

In this paper, we shall be concerned with the Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} = f(s), \tag{1.1}$$

where the sequence $\{\lambda_n\}$ increases and tends to infinity with n. Let $N(r)$ denote the number of λ_n which are less than r; then the number

$$D = \lim_{\xi \to 1} \{ \limsup_{r \to \infty} [N(r) - N(\xi r)]/[r - \xi r] \} \tag{1.2}$$

is called the *maximum density* of the sequence $\{\lambda_n\}$. Whenever a Dirichlet series is mentioned in this paper, it will always be assumed to have a sequence of exponents with finite maximum density. We shall be particularly interested in series of the form (1.1) which satisfy Ostrowski's gap condition; that is to say, series which are such that there exists an increasing sequence of integers $\{n_k\}$ and a positive constant $\mathcal{D}$, such that

$$\lambda_{n_k+1} - \lambda_{n_k} \geqslant \mathcal{D} \lambda_{n_k} \tag{1.3}$$

A Dirichlet series may converge at no finite point in the plane, it may converge at every finite point in the plane or else there may exist a finite number, σ_c such that the series converges in the half-plane $\operatorname{Re}(s) > \sigma_c$ but diverges at every point which has real part less than σ_c; no other case can occur. In the third-mentioned case, we may take $\sigma_c = 0$ without loss of generality. Let us then write

$$S_n(s) = \sum_{m=1}^{n} a_m e^{-\lambda_m s}; \tag{1.4}$$

we know that the sequence $\{S_n(s)\}$ cannot converge at any point outside the closure of the region of convergence, but it is possible that a subsequence, $\{S_{n_k}(s)\}$ may converge in a region D, larger than the region of convergence of (1.1); when this occurs, we say that (1.1) overconverges in D.

For power series, the phenomena of gaps and overconvergence are connected by the following well-known theorems of Ostrowski (see, for example Dienes [3]).

Theorem 1 (Ostrowski). *If a power series $\sum a_n z^n$ satisfies Ostrowski's gap condition, the sequence $\{\sum_1^{n_k} a_n z^n\}$ converges in some neighbourhood of each regular point on the circle of convergence of the series.*