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A metric result about the zeros of a complex polynomial ideal

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Introduction

Let us begin by listing some notations. We shall denote by K the field of complex numbers, by $K[x] = K[x^1, \ldots, x^n]$ a polynomial ring over K in n variables, and by K^n the n-dimensional vector space over K. The complex conjugation in K, and its natural extensions to $K[x]$ and K^n, will be indicated by the superscript ~ over the respective elements. Let $\gamma = (\gamma^1, \ldots, \gamma^n)$ be an element of K^n. It is called real if $\tilde{\gamma} = \gamma$, that is, if $\gamma^1, \ldots, \gamma^n$ are all real. The norm $\|\gamma\|$ of γ is defined as the non-negative number satisfying

$$\|\gamma\|^2 = \sum_{i=1}^{n} \tilde{\gamma}^i \gamma^i.$$

If, in $K[x]$, $f = f(x)$ is an element and $\mathfrak{a}$ an ideal, we denote by $d(\gamma; f)$ and $d(\gamma; \mathfrak{a})$ the distances in the sense of the norm between γ and the sets of complex zeros of f and of $\mathfrak{a}$ respectively. More precisely,

$$d(\gamma; f) = \inf\{\|\gamma - \gamma'\| \mid \gamma' \in K^n,\ f(\gamma') = 0\},$$
$$d(\gamma; \mathfrak{a}) = \inf\{\|\gamma - \gamma'\| \mid \gamma' \in K^n,\ f(\gamma') = 0 \quad \text{for every } f \in \mathfrak{a}\},$$

where the infimum of an empty set is counted as $+\infty$.

Now let $\mathfrak{a} = (f_1, \ldots, f_r)$ be an ideal of $K[x]$. There exists in $\mathfrak{a}$ a polynomial which has no more *real* zeros than the ideal $\mathfrak{a}$ itself, for

$$f = \sum_{\nu=1}^{r} \tilde{f}_\nu f_\nu$$

is clearly such a polynomial. The object of the present note is to prove a refinement of this result in the form of the following

Theorem. *Let $\mathfrak{a}$ be an ideal of $K[x]$. There exist a polynomial $f \in \mathfrak{a}$ and a positive constant c such that for every real $\alpha \in K^n$ we have*

$$d(\alpha; f) \geqslant c\, d(\alpha; \mathfrak{a}).$$

If $\mathfrak{a}$ has no complex zeros, $d(\alpha; \mathfrak{a}) = +\infty$ for every α, and the theorem gives the existence of an $f \in \mathfrak{a}$ without complex zeros, i.e. a non-zero constant polynomial. Thus in this case we have a form of Hilbert's "Nullstellensatz".