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# Differentiability properties of solutions of systems of differential equations

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## Introduction

Various algebraic characterizations of the differential equations with constant coefficients which only possess infinitely differentiable solutions have been given by Hörmander [4]. One of them is the following. The differential equation can be written

$$P(D)u=0, \tag{1}$$

where $P(\xi)=P(\xi_1, \ldots, \xi_\nu)$ is a polynomial and $P(D)$ is obtained by replacing $\xi_j$ by $-i\,\partial/\partial x^j$, and $u$ is a function of $x=(x^1, \ldots, x^\nu)$. Then, according to Theorem 3.7 in Hörmander [4], all (square integrable) solutions of (1) in a bounded domain $\Omega$ are infinitely differentiable functions (after correction on a null set) if and only if the set $V=\{\zeta; P(\zeta)=0\}$ satisfies the condition

$$\operatorname{Im}\zeta\to\infty \quad \text{when} \quad V\ni\zeta\to\infty. \tag{2}$$

An equivalent form of this condition is evidently that the distance from a real point $\xi$ to $V$ tends to infinity when $\xi\to\infty$. If (2) is fulfilled, it follows that every distribution $u$ satisfying a differential equation

$$P(D)u=f \quad \text{in} \quad \Omega, \tag{3}$$

where $\Omega$ is an open set and $f\in C^\infty(\Omega)$, is itself in $C^\infty(\Omega)$. This theorem is not explicitly stated in the quoted paper except when $f=0$ (cf. the end of section 3.5) but is an immediate consequence of formula (3.5.3) and the well-known properties of convolutions.

With a terminology, which has recently become generally accepted, the differential operator $P(D)$ and the polynomial $P(\zeta)$ are called *hypoelliptic* if the solutions of (3) are infinitely differentiable where this is true for $f$, or, which is equivalent, if (2) is fulfilled.

We shall here extend these results to systems of differential equations with constant coefficients. This extension is straightforward unless the system is *overdetermined*, that is, contains more equations than unknowns. In this case our proof is a simple consequence of the following theorem of Lech [5]:

*Let $I$ be an ideal of polynomials in $\zeta\in C_\nu$ with complex coefficients and $V_I$ the alge-*