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On homomorphisms and orthogonal systems

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1. Let G be a compact topological group, and let D be a subgroup of the group of all continuous homomorphisms T of G onto G. We define a class of functions A_D: $f \in A_D$ if for any two homomorphisms belonging to D

$$\int_G f(T_1 x)\overline{f(T_2 x)}\,dx = 0, \tag{1.00}$$

whenever $f(T_1 x)$ and $f(T_2 x)$ are different functions; and if

$$\int_G f(T_1 x)\overline{f(T_2 x)}\,dx = 1 \tag{1.01}$$

whenever $f(T_1 x)$ and $f(T_2 x)$ are identical. (We have especially $\int_G |f(x)|^2\,dx = 1$).

$$f(x) \text{ is continuous.} \tag{1.02}$$

We can also say that $\{f(Tx)\}$ is an orthonormal system.

We can now formulate the following problem: when does the fact that $f(x) \in A_D$ imply that $f(x)$ is a character or a simple combination of characters? We shall treat two special groups. In the first case, the group G is the topological group dual to the real line under the discrete topology. All functions $f(x) \in A_D$ are almost periodic, and Lemma 2 gives the tool we need for the proof of Theorem 1, which is the main result of the paper. In the second case, the group G is the unit circle under the usual topology. By using Theorem 1 we obtain a result for periodic functions with derivatives of all orders (Theorem 2). In this case we shall also investigate what happens when the orthonormal system $\{f(Tx)\}$ is complete (Theorem 3).

2. Let G be the topological group dual to the real line under the discrete topology [1], and let the group D consist of those homomorphisms of G, which, when $x \in G$ is real, have the form $Tx = ax$ (a is a real non-zero number). On the real line, the class of continuous functions on G is identical with the class of almost periodic functions, and we have

$$\int_G f(x)\,dx = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} f(x)\,dx = M\{f(x)\}.$$

We can now define this special class, which we shall call A_0, in a less abstract way. We say that the almost periodic function $f(x) \in A_0$, if