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Definitions of maximal differential operators

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Let $P(D)$ be a partial differential operator with constant coefficients and Ω an open set in R^ν (for notations and terminology cf. Hörmander [2], particularly pp. 176–177). In [2] the following two operators defined by $P(D)$ in $L^2(\Omega)$ were studied:

(i) The *minimal operator* P_0 defined as the closure of $P(D)$ in $C_0^\infty(\Omega)$.
(ii) The *maximal operator* P_w defined for those $u \in L^2(\Omega)$ such that $P(D)\,u = f$ in Ω in the distribution sense (or, which is the same thing, P_w is the adjoint of the minimal operator $\bar{P}_0$ defined by the formal adjoint $\bar{P}(D)$ of $P(D)$). P_w is also called the weak extension; it was denoted by P in [2].

There is an obvious lack of symmetry between these two definitions, one operator being defined by closure and one by duality. The reason for choosing the definition (ii) of the maximal operator is that it is important in some connections that the adjoint should be easy to study. On the other hand, it would sometimes be important to know that the maximal operator can be obtained by closing the operator $P(D)$ defined in a set of smooth functions. One might thus be interested in the following two "almost maximal" operators also:

(iii) The *strong extension* P_s which is the closure of $P(D)$ defined for those $u \in C^\infty(\Omega)$ such that $u \in L^2(\Omega)$, $P(D)u \in L^2(\Omega)$.
(iv) The *very strong extension* P_S which is the closure of $P(D)$ defined for those u which are restrictions to Ω of functions in $C_0^\infty(R^\nu)$.

It is obvious that we always have

$$P_S \subset P_s \subset P_w \tag{1}$$

and it is natural to expect that they should all be equal if sufficient regularity conditions are imposed on Ω. The following results are known previously:

(i) *For an arbitrary domain Ω we have $P_s = P_w$ if $P(D)$ is an operator of local type* ([2], Theorem 3.12).
(ii) *If $P(D)$ is homogeneous and not of local type (i.e. not elliptic modulo its lineality space) there always exists a domain Ω such that $P_s \neq P_w$.* (This has been proved by Schwarz [5] by extending an example given by the author.)

Concerning the very strong extension we shall prove here that $P_S = P_s = P_w$ if Ω is a bounded domain with a sufficiently smooth boundary (Theorem 2 be-