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A generalization of Picard's theorem[1]

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1. Introduction

1. Let $f(z)$ be meromorphic outside a closed point set E in the complex plane, and let $f(z)$ possess at least one singularity in E. If $f(z)$ cannot omit more than two values in the complement of E, we call E a *Picard set*.

By Picard's theorem, sets with only a finite number of points are Picard sets. In this paper, we shall generalize this result and show that all sufficiently thin countable sets with one limit point are also Picard sets.

Let E: $a_1, a_2, \ldots$ be a point set whose points converge to infinity. If a function $f(z)$, meromorphic outside E, is singular at some point a_ν, then of course $f(z)$ cannot omit more than two values. Hence, on studying whether E is a Picard set or not, we may restrict ourselves to functions $f(z)$ possessing their only singularity at infinity. In other words, we consider functions $f(z)$ meromorphic for $z \neq \infty$.

2. If $f(z)$ omits two values w_1 and w_2 in the whole finite plane, it is clear that $f(z)$ takes all other values outside E if the points of E tend to infinity with sufficient rapidity. For $f(z)$ is then at least of order 1, and this implies an upper bound for the velocity with which for any $w \neq w_1, w_2$, the w-points converge towards infinity.

If however, $f(z)$ omits only one value or none at all for $z \neq \infty$, no similar conclusions can be drawn. For it is possible to construct entire or meromorphic functions for which all w-points tend to infinity as rapidly as we please. Removing from the plane the w-points for three different values w (for two values for entire functions), we obtain sets E which are certainly not Picard sets.

The following example shows that in such a case, even the distance of any two points of E can be made arbitrarily large. Put

$$f(z) = \prod_{\nu=1}^{\infty} \frac{1 - z/b_\nu}{1 + z/b_\nu}, \tag{1}$$

where $b_\nu > 0$, $b_1 < b_2 < \ldots$, and b_ν tends rapidly to infinity. Clearly, $f(z)$ has its zeros at $z = b_\nu$, poles at $z = -b_\nu$, and 1-points on the imaginary axis. If the 1-points are denoted by $\pm i c_\nu$, $c_0 = 0, c_\nu > 0, \nu = 1, 2, \ldots,$ we have the identity

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