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Uniform approximation with Diophantine side-conditions of continuous functions

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This article is composed of two entirely different parts. The first part deals with approximation of real continuous functions in one variable by rational functions whose zeros and poles all belong to certain prescribed sets. The second part is about approximation of real continuous functions in several variables by polynomials with integral coefficients. Two general theorems are proved and finally some special results are discussed. Fekete has completely analysed the problem in one variable [3].

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Theorem 1: *Each positive continuous function $f(x)$ admits uniform approximation on $I=[-1,+1]$ by the quotient of two real polynomials, whose zeros belong to two given sets P and $\bar{P}$ satisfying:*

1) $\bar{P}$ is the image of P under reflection in the real axis.

2) P (and $\bar{P}$) is situated on a line segment whose extension does not intersect I perpendicularly.

3) P (and $\bar{P}$) has a subset P' (and $\bar{P}'$) which is dense in itself (i.e. $P' \subset$ the set of all the accumulation points of P').

Proof: We use the following well-known theorem: The set M is dense in $C(-1,1)$ if and only if each linear functional in $C(-1,1)$ which is zero for every element of M is identically zero. A real functional $L(f)$ in $C(-1,1)$ may be written as

$$L(f)=\int_{-1}^{+1} f(x)\,d\mu(x)$$

where μ is a real function of bounded variation. We define M as the set of functions $\{e^{i\varphi}/(x-a)+e^{-i\varphi}/(x-\bar{a})\}$ for all $a=\alpha+i\beta\in P'$; φ is the argument of the line l to which a belongs. The assumption $L(f)=0$ for $f\in M$ then implies that the function

$$u(\alpha,\beta)=\int_{-1}^{+1}\left(\frac{e^{i\varphi}}{x-a}+\frac{e^{-i\varphi}}{x-\bar{a}}\right)d\mu(x)$$

vanishes for all $a\in P'$. But $u(\alpha,\beta)$ being the real part of an analytic function $\psi(a)$, holomorphic for $a\notin I$, is harmonic, and we consequently have $u(\alpha,\beta)=0$ on the whole line l except in its possible intersection with I. Using the principle of reflection we get: