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The set of extreme points of a compact convex set

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0.1. Introduction

The purpose of this note is to establish necessary and sufficient conditions for a set S in a Fréchet space X to be the set of extreme points of some compact convex set.[1]

We shall first study the case where X is a Euclidean space R^n. Suppose that S is the set of extreme points of some compact convex set K. As is well known, S need not be closed. But it is also well known that K must be the convex hull of S and, consequently,

the closure of S is contained in the convex hull of S. (b)

Further obvious necessary properties of S are that

the closure of S is compact (a)

and that

no point of S is a barycenter of a finite set of other points of S. (c′)

The following is obviously an equivalent formulation of (c′): For all $A \subset S$, no point of S is in the convex hull of A without being in A, or, with obvious notation,

$$S \cap H(A) \subset A \ \textit{for all}\ A \subset S. \tag{c}$$

Thus, in a Euclidean space, conditions (a, b, c) are necessary. In Theorem 2.1 they are proved to be sufficient. We shall also prove that some apparently stronger conditions are necessary.

When X is a general Fréchet space (Theorem 3.1), we have to make some changes in conditions (b) and (c), for it develops that (b) is not necessary and (a, b, c) are not sufficient. However, if the convex hull occurring in (b) and (c) is replaced by a certain larger "hull", the resulting conditions will be necessary and sufficient. This new hull H_C of a set S was considered by Choquet [4, 5, 6]. It is defined as the set of barycentra of those positive Radon measures μ of total mass one on $\bar{S}$ which are contained in S in the sense that $(\bar{S}-S)$ has μ-measure zero. Choquet [5 or 6] proved that each compact convex set K in a Fréchet space is the H_C-hull of its set of extreme points. The Krein-

[1] For terminology, see [1] and [2].