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Sums and products of commuting spectral operators[1]

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1. Introduction

Given two spectral operators T_1 and T_2, on a complex Banach space $\mathfrak{X}$, it is interesting to know if T_1+T_2 and T_1T_2 are spectral operators too. This problem is treated in [3] and [4]. It is proved there that if the space $\mathfrak{X}$ is weakly complete, and the operators T_1 and T_2 commute and the Boolean algebra of projections generated by the resolutions of the identity of T_1 and T_2 is bounded, then both T_1+T_2 and T_1T_2 are spectral operators. Moreover, if $T_1=S_1+N_1$, $T_2=S_2+N_2$, where S_1 and S_2 are scalar operators and N_1, N_2 are generalized nilpotents and S_1, N_1, S_2, N_2 commute, then S_1+S_2 and S_1S_2 are scalar operators and N_1+N_2, $S_1N_2+S_2N_1+N_1N_2$ are generalized nilpotents. The main problem in this paper will be to determine the resolutions of the identity of T_1+T_2 and T_1T_2. By the above remark it is enough to consider the case where T_1 and T_2 are scalar operators. A second problem treated here is to find the poles of the resolvents of T_1+T_2 and T_1T_2. In this part we do not assume that T_1 and T_2 are spectral operators.

2. Notation

We use here the notation and definitions of [3]. Let $\mathfrak{X}$ be a complex Banach space. A spectral measure is a set function $E(\cdot)$ defined on Borel sets in the complex plane whose values are projections on $\mathfrak{X}$ which satisfy:

1. For any two Borel sets σ and δ, $E(\sigma)E(\delta)=E(\sigma\cap\delta)$.
2. Let ϕ be the void set and P the complex plane, then

$$E(\phi)=0 \quad \text{and} \quad E(P)=I.$$

3. There exists a constant M such that $|E(\sigma)|\leq M$, for every Borel set σ.
4. The vector valued set function $E(\cdot)x$ is countably additive for each $x\in\mathfrak{X}$.

T is a spectral operator whose resolution of the identity is the spectral measure $E(\cdot)$ if

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