ARKIV FÖR MATEMATIK Band 3 nr 40

Communicated 24 April 1957 by LENNART CARLESON

Metric criteria of normality for complex matrices of order less than 5

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I. Introduction

We denote a (finite-dimensional) complex Hilbert space by F. Its elements (vectors) are denoted f, g and the scalar product of $f, g \in F$ is written (f, g). The norm of $f \in F$ is $(f, f)^{\frac{1}{2}} = \|f\|$. Elements (matrices) of the algebra $B(F)$ of endomorphisms on F are denoted by capital letters other than B and F. The norm of $A \in B(F)$ is defined by $\|A\| = \sup_{f \in F} \|Af\| \cdot \|f\|^{-1}$. The adjoint A^* of A is defined by $(Af, g) = (f, A^* g)$ for all $f, g \in F$.

An element A of $B(F)$ is called normal if it commutes with its adjoint: $A^* A = A A^*$.

As is well known, $A \in B(F)$ is normal if and only if it can be written as a sum

$$A = \sum_1^m \lambda_k E_k, \tag{I.1}$$

where λ_k are complex scalars and $E_k \in B(F)$ satisfy the conditions

$$\sum_1^m E_k = I; \qquad E_j E_k = 0, \quad j \neq k; \qquad E_k = E_k^* = E_k^2. \tag{I.2}$$

The set $\operatorname{sp} A = \{\lambda_k \mid E_k \neq 0\}$ is called the spectrum of A. From eqs. (I.1) and (I.2) it is easy to conclude that for all polynomials $p(t)$ in one variable t with complex coefficients one has

$$\|p(A)\| = \max_{\lambda \in \operatorname{sp} A} |p(\lambda)|. \tag{I.3}$$

According to a theorem of v. Neumann [1], the following converse of (I.3) holds true. If Γ is a finite subset of the complex plane and

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[1] This article was written at the Department of Mathematics, Royal Institute of Technology, Stockholm, while the author was holding a scholarship from the State Council for Technical Research.