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A non-closed relative spectrum

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Let E be a linear topological space, T a continuous linear transformation from E into itself, I the identity transformation on E and put $T_\lambda = T - \lambda I$. If, for a complex λ, there exists a continuous linear transformation R_λ from E into E, which satisfies the condition $T_\lambda R_\lambda T_\lambda = T_\lambda$, we say that T_λ is relatively regular for this value of λ and that R_λ is the relative inverse of T_λ. Those values of λ for which T_λ is not relatively regular constitute the relative spectrum of T. The relative spectrum is contained in the intersection of the right spectrum and the left spectrum of T. Unlike this set, however, the relative spectrum need not be a closed set even if E is complete and metrizable as we will show by an example. (A statement to the contrary has appeared in the literature and has not, to my knowledge, been disproved earlier.) Our tool in constructing this example will be the following proposition.

Let the complex number α belong to the relative spectrum of the linear continuous transformation T from E into E. Then the linear continuous transformation T' from the direct sum $E+E$ of E with itself into $E+E$, represented by the (block) matrix

$$T' = \begin{pmatrix} \alpha I & 0 \\ I & T \end{pmatrix},$$

where I denotes the identity transformation in E, has for relative spectrum the relative spectrum of T with exception of the point α.

Firstly, we show that α does not belong to the relative spectrum of T'. This is settled by the identity

$$T' - \alpha I' = \begin{pmatrix} 0 & 0 \\ I & T - \alpha I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ I & T - \alpha I \end{pmatrix} \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ I & T - \alpha I \end{pmatrix}.$$

Secondly, we show that the points of the relative spectrum of T other than α belong to the relative spectrum of T'. Suppose that $R'_\lambda = \begin{pmatrix} A_\lambda & B_\lambda \\ C_\lambda & D_\lambda \end{pmatrix}$ is a relative inverse of $T'_\lambda = T' - \lambda I'$ for one of these values of λ. The identity $T'_\lambda R'_\lambda T'_\lambda = T'_\lambda$ becomes

$$\begin{pmatrix} (\alpha-\lambda)^2 A_\lambda + (\alpha-\lambda) B & (\alpha-\lambda) B_\lambda (T-\lambda I) \\ (\alpha-\lambda)(A_\lambda + (T-\lambda I) C_\lambda) + B_\lambda + (T-\lambda I) D_\lambda & B_\lambda (T-\lambda I) + (T-\lambda I) D_\lambda (T-\lambda I) \end{pmatrix} = \begin{pmatrix} (\alpha-\lambda) I & 0 \\ I & T-\lambda I \end{pmatrix}. \quad (1)$$