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On the connection between Hausdorff measures and capacity

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1. The metrical characterization of pointsets has been carried out along two different lines. Hausdorff (1919) introduced what is now called Hausdorff measures and the concept of capacity was first given a general sense by Polya–Szegö (1931).[1] The first general result on the connection between the two concepts was given by Frostman [3] (1935). He proved that if a closed set has capacity zero, then its Hausdorff measure vanishes for every increasing function $h(r)$, $h(0)=0$, such that

$$\int_0^1 \frac{h(r)}{r}\,dr < \infty. \tag{1}$$

It has since then been an open question whether or not a converse of this result holds true: given a closed set E of positive capacity, does there exist a measure function $h(r)$ such that (1) holds and such that the corresponding Hausdorff measure is positive? This is known to be true e.g. for Cantor sets. The main object of this note is to exhibit a set E for which it fails to hold. This will make it clear that the two ways of measuring sets E are fundamentally different.

In the other direction it has been proved by Erdös and Gillis [2] that if a set E has finite Hausdorff measure with respect to $(\log(1/r))^{-1}$, then its capacity vanishes. We shall give a new and very simple proof of this result. The method will also permit us to prove, for sets of positive capacity, the existence of a uniformly continuous potential, a result that does not seem to have been observed before.

2. Let I be a subinterval of $(0,1)$. By $(m,q)\,I$, m an integer, we denote a subdivision of I into smaller intervals in the following way. The subintervals cover I and have lengths (from left to right): $e^{-m}, e^{-q}, e^{-m-1}, e^{-q}, \ldots, e^{-q}, e^{-2m}$. We assume that m and q are so chosen that this actually gives a covering of I, and we speak of the m-intervals and the q-intervals. We shall construct E applying this kind of subdivision on intervals, and we shall each time let the m q-intervals of length e^{-q} belong to the complement of E.

Let us assume that we have applied the above method n times and in this way obtained the set E_n of m-intervals. Let μ_n be a distribution of unit mass with constant density on each interval of E_n and let $u_n(x)$ be the corresponding potential. Let I be the interval of E_n to be subdivided. We distribute the mass $\mu_n(I)/(m+1)$ uni-

[1] For definitions see [4], pp. 114 ff.