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A prediction problem in game theory

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1 Diagram

1. Introduction

Two persons, X and P, play the following game. X controls a random mechanism producing a stationary stochastic process $x=\{x_t\}$. The mean value $E\,x(t)=0$ and the variance $E\,x^2(t)=1$ are given, but the form of the spectrum of the process can be determined completely by X. The player P observes the values $x(t)$ for $t\le 0$ and wants to predict the value of

$$z=\int_0^1 a(t)\,x_t\,dt=A\,x,$$

where $a(t)$ is a real and continuous function of t in (0, 1). For this purpose P has available all the linear predictors formed from the stochastic variables x_t, $t\le 0$. Let us denote the chosen predictor by p and the predicted value by $p\,x$.

The payoff function of the game is $(A\,x-p\,x)^2$. X wishes to maximize the quantity $\|A\,x-p\,x\|^2=E\,[A\,x-p\,x]^2$ and P tries to minimize the same expression.

In the next section we study the value of $\max\min\|A\,x-p\,x\|^2$ and how this value is attained. In section 3 we show that this value coincides with $\min\max\|A\,x-p\,x\|^2$, the game is definite, and the pair of pure strategies is obtained that forms a solution of the game. It may be a matter of convention whether the strategy of the player X is considered as pure (a single choice of $g(t)$) or randomized (a choice of the stochastic process x). Sections 4 and 5 are devoted to some applications of the theorem in section 3.

The author was led to this result when working on an applied problem. The reader may be interested in consulting the paper of M. C. Yovits and J. L. Jackson [1] for a different approach to a similar problem.

2. Derivation of max min

When P first chooses his strategy p to minimize $\|A\,x-p\,x\|^2$ and then X chooses x to maximize the minimum value, it is clear that we can immediately limit the set of strategies of X to processes that are completely non deterministic.