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On the associativity formula for multiplicities

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This paper is concerned with a theorem of Chevalley on multiplicities in a local ring ([1], Theorem 5, p. 25). We shall present a generalized form of this theorem, for which we can give a new and rather simple proof. Before stating our theorem we introduce some notations. If $\mathfrak{q}$ is a primary ideal belonging to the maximal ideal of a local ring, then $e(\mathfrak{q})$ means its multiplicity, defined according to Samuel, and $L(\mathfrak{q})$ its length; if $\mathfrak{a}$ is an arbitrary ideal† in a Noetherian ring R and $\mathfrak{p}$ a minimal prime ideal of $\mathfrak{a}$, then we define $e(\mathfrak{a}; \mathfrak{p}) = e(\mathfrak{a} R_{\mathfrak{p}})$ and $L(\mathfrak{a}; \mathfrak{p}) = L(\mathfrak{a} R_{\mathfrak{p}})$, where $R_{\mathfrak{p}}$ denotes the generalized ring of quotients with respect to $\mathfrak{p}$. It may be pointed out here that our result depends in an essential way on Samuel's notion of multiplicity, which is more general than Chevalley's original notion.

Our theorem reads:

Theorem 1. *Let Q be a local ring of dimension r and let $\{x_1, \dots, x_r\}$ be a system of parameters in Q. Put $\mathfrak{q} = (x_1, \dots, x_r)$ and $\mathfrak{o} = (x_{m+1}, \dots, x_r)$, where $0 \leq m \leq r$. Let $\mathfrak{p}$ range over those minimal prime ideals of $\mathfrak{o}$ for which* $\dim \mathfrak{p} + \operatorname{rank} \mathfrak{p} = \dim Q$. *Then*

$$e(\mathfrak{q}) = \sum_{\mathfrak{p}} e((\mathfrak{q} + \mathfrak{p})/\mathfrak{p})\, e(\mathfrak{o}; \mathfrak{p}).$$

Chevalley's theorem is restricted to local rings which admit a nucleus. (It is formulated for the even smaller class of geometric local rings.) In his theorem, $\mathfrak{p}$ ranges over all minimal prime ideals of $\mathfrak{o}$. This difference from our theorem comes from the fact that in a local ring which admits a nucleus it is true for every prime ideal $\mathfrak{p}$ that $\dim \mathfrak{p} + \operatorname{rank} \mathfrak{p} = \dim Q$. Chevalley's theorem as well as ours has its greatest importance in the algebro-geometric theory of intersection-multiplicities.

We begin our proof by deriving a certain expression for the multiplicity of an ideal generated by a system of parameters (Theorem 2, Section 1). Theorem 1 is then proved by induction on the dimension of Q (Sections 2 and 3). The proof is based directly on the fundamental properties of Noetherian rings and of local rings.‡ The local rings which occur during the demonstrations are

† By an ideal we shall always mean a proper ideal; in other words, the whole ring does not count as an ideal.

‡ As a general reference, also for the terminology, see [2].