ARKIV FÖR MATEMATIK Band 3 nr 26

Communicated 9 November 1955 by OTTO FROSTMAN

A note on a problem of Boas

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Boas has in a paper [1] generalized certain theorems of Plancherel and Pólya [2] concerning simultaneous convergence of

$$\int_{-\infty}^{+\infty} |f(x)|^p \, dx \quad \text{and} \quad \sum_{-\infty}^{+\infty} |f(\lambda_n)|^p$$

for entire functions $f(z)$ of exponential type. He also puts the question how to treat corresponding problems for functions regular in a half-plane, especially what may be precisely stated as follows.

Problem. *Let $f(z)$ be regular for $x \geq 0$, $z = x + iy$, and such that, if $z \to \infty$ in this half-plane,*

$$\limsup \frac{\log |f(z)|}{|z|} = c, \qquad 0 < c < \infty. \tag{1}$$

Let $\varphi(t)$ be, for $t \geq 0$, a non-decreasing convex function of $\log t$ with $\varphi(0) = 0$.

Consider further a sequence of positive numbers $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ such that

$$\inf_n (\lambda_{n+1} - \lambda_n) \geq 2\delta > 0. \tag{2}$$

Does then

$$\int_0^\infty \varphi\{|f(x)|\} \, dx < \infty \tag{3}$$

imply

$$\sum_0^\infty \varphi\{e^{-c\delta} |f(\lambda_n)|\} < \infty? \tag{4}$$

At first I thought the answer ought to be negative. Therefore the following affirmative proof was not finally elaborated until I heard Professor Lennart Carleson express a contrary opinion: that my original condition (5) was superfluous.

By means of an example Boas shows that the factor $e^{-c\delta}$ in (4) cannot be dropped. It signifies a number arbitrarily close to 1 since—as soon as (2) is fulfilled—δ may be chosen arbitrarily close to 0.

The proof is given in two steps.