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Some groups of order $p^r q^s$ with Abelian subgroups of order p^r contained in the central

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The group of order $p^r q^s$ where p and q are different prime numbers may be generated by $A_i B_j$ where A_i runs through all elements of a subgroup of order p^r and B_j all elements of a subgroup of order q^s. There are $p^r q^s A_i B_j$ and they are all different. Hence they exhaust the group $G_{p^r q^s}$. (Here and in the following "G_n" denotes a group of order n.) This means that if under certain conditions every A_i must be permutable with every B_j and if a pair of groups, G_{p^r}, G_{q^s}, fulfils these conditions, there is one and only one group of order $p^r q^s$ with just these groups as subgroups, because the relations between A_i and B_j are completely determined in this case. If under these conditions one of the groups in the pair, say G_{p^r}, is Abelian, G_{p^r} is contained in the central of the group $G_{p^r q^s}$.

It has been shown that if $p > q^s$ and $p \not\equiv 1 \pmod q$ and G_{p^r} is a cyclic subgroup of $G_{p^r q^s}$, then G_{p^r} must be contained in the central of $G_{p^r q^s}$. This also means that there can only be as many abstract groups of a given order $p^r q^s$ with these conditions fulfilled as there are different groups of order q^s.[1] In the following the case where G_{p^r} is an Abelian group generated by two elements will be considered and the theorem to be deduced is:

Theorem: *If G_{p^r} is an Abelian subgroup of $G_{p^r q^s}$ generated by two elements of different order and $p > q^s$ and $p \not\equiv 1 \pmod q$, or if G_{p^r} is an Abelian subgroup of $G_{p^r q^s}$ generated by two elements of the same order and $p > q^s$ and $p^2 \not\equiv 1 \pmod q$, then G_{p^r} must be contained in the central of $G_{p^r q^s}$.*

The proof requires some lemmas.

Lemma 1. When $p > q^s$, there is only one subgroup of order p^r of the group $G_{p^r q^s}$.

Suppose G_{p^r} and G'_{p^r} were two different subgroups of $G_{p^r q^s}$. Then G'_{p^r} would contain at least some element, say A', not contained in G_{p^r} and of order p^v, where $v \neq 0$. $(A')^n A_i$ would then produce p^{r+1} different elements, when n takes the values $1, 2, \ldots, p$, and A_i runs through all elements of G_{p^r}. They are all different, because if $(A')^m A_i = (A')^n A_j$ we would have $(A')^{m-n} = A_j A_i^{-1}$ and A' would be an element of G_{p^r} if $m \neq n$, contrary to the assumptions, because in this case $m - n \not\equiv 0 \pmod p$.

[1] E. GÖTLIND: Några satser om grupper av ordningen $p^r q^s$. (Some lemmas about groups of order $p^r q^s$.) *Norsk Matematisk Tidsskrift*, 1948, p. 11, together with a correction note to this paper: Not till uppsatsen "Några satser om grupper av ordningen $p^r q^s$", the same journal, 1949, p. 59.