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On the Diophantine equation $x^2+8D=y^n$

By TRYGVE NAGELL

§ 1.

In a previous paper[1] I showed that the Diophantine equation

$$x^2+8=y^n \quad (n \geq 3) \tag{1}$$

has no solution in positive integers x and y when n is not a prime $\equiv \pm 1 \pmod 8$. If n is a prime $\equiv \pm 1 \pmod 8$, there is at most one solution in positive integers.

It is, however, possible to obtain the following improvement of this result:

Theorem 1. *The Diophantine equation* (1), *where n is an integer ≥ 3, has no solution in positive integers x and y.*

The proof will be given in § 5.

In this paper we shall examine the more general equation

$$x^2+8D=y^n, \tag{2}$$

where D is a square-free, odd integer ≥ 1, and where n is an integer ≥ 3.

We begin by proving the following lemma:

Lemma 1. The equation (2) *has no solution in even integers x and y if $n \geq 4$. If $n=3$ and if the number of ideal classes in the quadratic field $\boldsymbol{K}(\sqrt{-2D})$ is not divisible by* 3, *the equation* (2) *is solvable in even integers x and y only when $D=6a^2 \mp 1$, a integer; corresponding to this value of D there is the single integral solution $y=16a^2 \mp 2$.*

Proof. Let x, y be a solution of (2) in integers. If x is even, y is so. Then y^n is divisible by 8. Hence by (2) x is divisible by 4. Since D is odd, y^n must be divisible by exactly 8, and this implies $n=3$. If we put $x=4x_1$ and $y=2y_1$, we get

$$(2x_1)^2+2D=2y_1^3. \tag{3}$$

The ideal factors $(2x_1+\sqrt{-2D})$ and $(2x_1-\sqrt{-2D})$ of the left-hand side have the greatest common divisor $(2, \sqrt{-2D})$. Hence it follows from (3)

[1] See NAGELL [1], § 2. Figures in [ ] refer to the Bibliography at the end of this paper.