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The reality of the eigenvalues of certain integral equations

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With 4 figures in the text

§ 1. Introduction

In this paper we shall study the reality of the eigenvalues in some integral equations of the Fredholm type

$$\varphi(x) = \lambda \int_0^1 K(x, y)\, \varphi(y)\, dy.$$

The kernel $K(x, y)$ is assumed to be 0 above a certain curve in the square $0 \leq {x \atop y} \leq 1$ where it is defined. Below the curve we suppose that $K(x, y) = P(x)\, Q(y)$. Let the curve have the equation $y = f(x)$ and make the following assumptions:

(α) $f(x)$ is non-decreasing,

(β) $\lim\limits_{t=+0} f(x-t) > x$ except possibly for $x = 0$ and $x = 1$,

(γ) $P(x)\, Q(x)$ is integrable in $0 \leq x \leq 1$.

We shall study two types of kernels:

Kernel A: The curve does not pass through $(0, 0)$ nor through $(1, 1)$ (fig. 1).
Kernel B: The curve goes through $(0, 0)$ or $(1, 1)$ or both points (fig. 2).

In [1] I have obtained explicit expressions for the corresponding denominators of Fredholm. In equation A they are polynomials in λ of degree depending only on the curve $y = f(x)$. I shall give an account of the formulas in question.

Let $f^2(x)$ mean $f(f(x))$, generally $f^n(x)$ the nth iterated function. We also introduce the in an appropriate way defined inverse $f^{-1}(x)$ which we give the value 0 for $0 \leq x \leq f(0)$. In the integral equation A, restricted to the square $0 \leq {x \atop y} \leq \alpha$, the denominator of Fredholm becomes:

$$D(\alpha, \lambda) = 1 - \lambda F_1(\alpha) + \lambda^2 F_2(\alpha) - \cdots + (-\lambda)^n F_n(\alpha), \tag{1}$$