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Notes on the Diophantine equation $y^2 - k = x^3$

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The purpose of this article is to give some corrections and additions to my dissertation "On the Diophantine equation $y^2 - k = x^3$" (Uppsala 1952). In the following I denote that paper by H and refer to the bibliography given there as e.g. (H, [4]). In fact I have succeeded in solving all the equations with $0 < k \leq 100$.

The problem of solving an equation $y^2 - k = x^3$ is equivalent to solving a finite number of equations $(a, b, c, d) = 1$, where (a, b, c, d) is a binary cubic form. To be short I name this form soluble and (u, v) a solution of the form, if there is any integer solution (u, v) of $(a, b, c, d) = 1$. In the first part of H it is shown how to determine those equations and further their solubility is discussed. A soluble form may be written

$$F(u, v) = u^3 + p u^2 v + q u v^2 + r v^3 = (1, p, q, r) \tag{1}$$

and corresponds to a cubic ring $R(\theta)$, where

$$F(\theta, -1) = \theta^3 - p\theta^2 + q\theta - r = 0. \tag{2}$$

If $k > 0$, the form $F(u, v)$ has always a negative discriminant. Then every integer solution of (1) corresponds to a unit of the type

$$\varepsilon^n = u + v\theta \tag{3}$$

and vice versa, where ε is the fundamental unit of the ring $R(\theta)$. Hence the decisive question is to determine all such units (3).

If $0 < \varepsilon < 1$, the case $n < 0$ can easily be examined by H, Lemma 7, p. 25. This lemma was inserted a short time before the printing and hence it is not applied all through. Further, for want of space, the proof was too short. Hence I repeat the lemma here with a detailed proof:

Lemma 7. *A soluble irreducible form can always by a unimodular substitution be written* $F(u, v) = (1, p, q, r)$, *where* $p \leq 1$ *and* $r > 0$. *Suppose* $D(F) < 0$ *and* $0 < \varepsilon < 1$, *where* ε *is the fundamental unit in the corresponding cubic ring. Then, if* v_1 *and* v_2 *are positive integers and if* $D\left(1, p, q, r - \frac{1}{v_1^3}\right)$ *and* $D\left(1, p, q, r + \frac{1}{v_2^3}\right)$