ARKIV FÖR MATEMATIK Band 3 nr 2

Read 14 October 1953

On a special class of Diophantine equations of the second degree

By Trygve Nagell

§ 1. Ambiguous ideals in real quadratic fields and Diophantine equations

Given the square-free integer $D>1$, the determination of the ambiguous ideal classes in the real quadratic field $\boldsymbol{K}(\sqrt{D})$ depends essentially on the following fact:

Theorem 1. *Let ε be the fundamental unit in $\boldsymbol{K}(\sqrt{D})$, and let*

$$\mathfrak{a}_1,\ \mathfrak{a}_2,\ \ldots,\ \mathfrak{a}_s$$

be all possible products of different ambiguous prime ideals in $\boldsymbol{K}(\sqrt{D})$.

If $N(\varepsilon)=-1$, none of the ideals $\mathfrak{a}_i$ is principal, apart from $(\sqrt{D})$.

If $N(\varepsilon)=+1$, exactly two of the ideals $\mathfrak{a}_i$ are principal, apart from $(\sqrt{D})$. The product of these principal ideals is $=(2\sqrt{D})$ when D is odd and the norms of the ideals are even; in all other cases the product is $=(\sqrt{D})$.

See f. ex. Hilbert [1], § 75 and Hecke [2], § 45.[1]

This theorem may also be formulated as follows:

Theorem 2. *Let D be a given square-free integer >1, and let C be any square-free divisor of $2D$, such that $C \neq 1$ and $\neq \pm D$. When $D \equiv 1 \pmod 4$, C shall be odd.*

Part 1. If the Diophantine equation

$$u^2 - D v^2 = C \tag{1}$$

is solvable in integers u and v for $C=-1$, it is not solvable for any other value of C.

If it is not solvable for $C=-1$, it is solvable for exactly two different values of C. The product of these two values of C is $=-4D$ when D is odd and C is even; in all other cases the product is $=-D$.

[1] Figures in [ ] refer to the Bibliography at the end of this paper.