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Uniform bound for Hecke L-functions

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1. Introduction

Our principal aim in the present article is to establish a uniform hybrid bound for individual values on the critical line of Hecke L-functions associated with cusp forms over the full modular group. This is rendered in the statement that for $t\geqslant 0$,

$$H_j\left(\tfrac{1}{2}+it\right) \ll (\varkappa_j+t)^{1/3+\varepsilon}, \tag{1.1}$$

$$H_{j,k}\left(\tfrac{1}{2}+it\right) \ll (k+t)^{1/3+\varepsilon}, \tag{1.2}$$

with the common notation to be made precise in the course of discussion.

Most of the arithmetically significant Dirichlet series, such as the Riemann zeta-function $\zeta(s)$, Dirichlet L-functions, and Hecke L-functions associated with various cusp forms, satisfy Riemannian functional equations connecting values at $s=\sigma+it$ and $1-s$ of the respective functions. Essentially best possible estimates for these functions near the lines $\sigma=1$ and $\sigma=0$ can usually be deduced from the definition of the respective functions and their functional equations. From this, bounds in the critical strip $0<\sigma<1$, in particular on the critical line $\sigma=\frac{1}{2}$, follow readily via the Phragmén–Lindelöf convexity principle; thus they are called *convexity bounds*. In general, there is a quantity $B(g,t)$ characterising the size of a function $g\left(\frac{1}{2}+it\right)$ of the above kind in a given t-range in such a way that the convexity bound is stated as

$$g\left(\tfrac{1}{2}+it\right) \ll B(g,t)^{1/2+\varepsilon}, \quad t>0, \tag{1.3}$$

with the usual usage of the symbol ε (see Convention 1 at the end of this section). For instance, $B(\zeta,t)=t^{1/2}$, or perhaps more naturally $B(\zeta^2,t)=t$. In view of the generalised

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